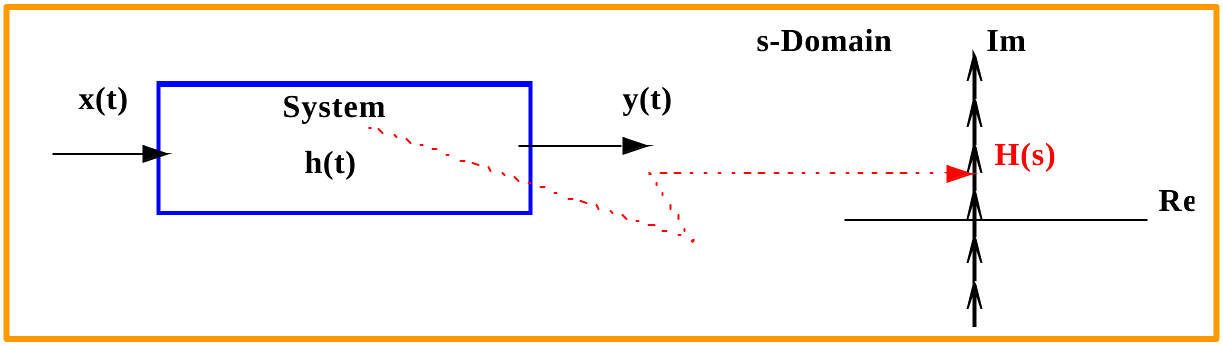
Chapter on Sinusoidal Frequency Response of Systems

4.1 S-Domain Responses

If the input to a continuous system is a sinosoid and we wait all the transient responses die to bring the system to a *steady-state*, the output from the system is now called the steady-state response. If we map this response onto the imaginary axis in s-plane we get the *frequency response* of the system.



$$x(t) = A.Cos(wt + q_0) \qquad (4.1)$$

Under steady-state conditions, the response is also a sinusoid:

$$y_{SS}(t) = A.|H(jw)|.Cos(wt + q_0 + j) = A.\left|H(s)\right|\Big|_{s=jw}.Cos(wt + q_0 + j) \qquad (4.2)$$

where the subscript *ss* stands for steady-state and we have implicitly defined the frequency response by:

$$H(jw) = H(s)\Big|_{s=jw}$$

We can now define the magnitude and phase responses:

$$|H(jw)| = M \quad \text{and} \quad j = \arg(H(jw)) = \angle H(jw) = P \qquad (4.3)$$

In the s-domain the response would be:

$$Y_{SS}(s) = H(s).X(s) \qquad (4.4)$$

where

$$H(jw) \equiv H(s)\Big|_{s=jw} = \frac{\sum_{k=0}^{L} b_k (jw)^k}{a_0 + \sum_{k=1}^{N} a_k (jw)^k} \qquad (4.5)$$

Properties of Frequency Response:

1. Amplitude or Magnitude: $M = |H(jw)| = |H(-jw)| = \sqrt{a^2 + b^2}$ (4.6)
2. In Decibels: $M_{dB} = 20\log_{10}(M)$ (4.7)

3. Phase: $$P = \arg(H(jw)) = \angle H(jw) = -\angle H(-jw) = \arctan(b/a) \qquad (4.8)$$

4. Bandwidth: Let M_{max} be the maximum value of the amplitude (magnitude) in the range of frequencies: $w_1 \le w \le w_2$ for which the amplitude is limited to: $0.707 M_{\max} \le M \le M_{\max}$ then this frequency range is called the bandwidth: $BW = w_2 - w_1$. In the dB scale, we measure the bandwidth where the amplitude response drops to -3.0 dB from its peak value of 1.0.

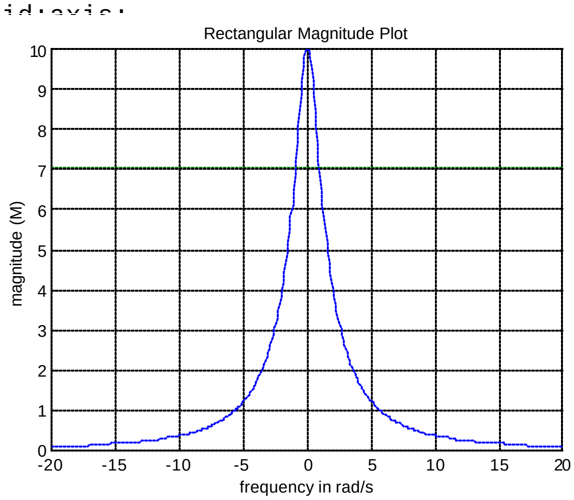
Frequency Response Plots:

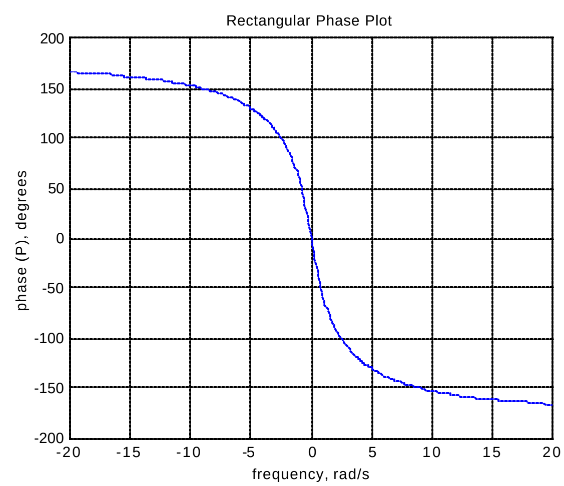
1. **Rectangular Plots:** They are the plots of M versus jw and P versus jw, respectively.

Example 4.1: Let us obtain the rectangular plots for the following transfer function:

$$H(s) = \frac{40}{s^2 + 11s + 10} = \frac{40}{(s+1)(s+10)}$$

```
%Example 4.1 Rectangular plots of Frequency Responses
a=[1,5,4];            % denominator coefficients in descending order
b=[40];                  % numerator coefficient
w=-20:.1:20;          % initial frequency:increment in rad/s:final frequency
H=freqs(b,a,w);          % frequency response call in Matlab
mag=abs(H);
g=0.707*ones(size(w))*max(mag); % line to show bandwidth
axis([-20,20,0,1]);
plot(w,mag,w,g); title('Rectangular Magnitude Plot');
xlabel('frequency in rad/s'); ylabel('magnitude (M)');
grid; pause;
phase = angle(H); phase = phase*180/pi;% changes phase from radians to degrees
axis([-20,20,-180,180])
plot(w,phase),title('Rectangular Phase Plot');
xlabel('frequency, rad/s');ylabel('phase (P), degrees');
grid;axis;
```

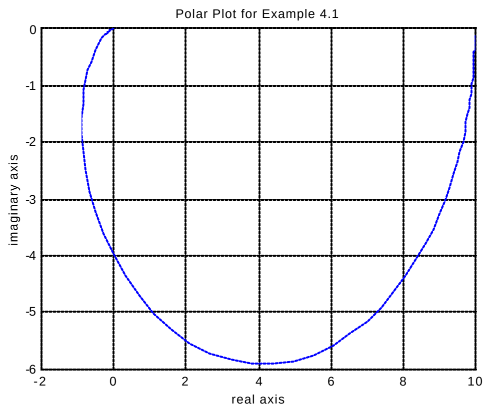


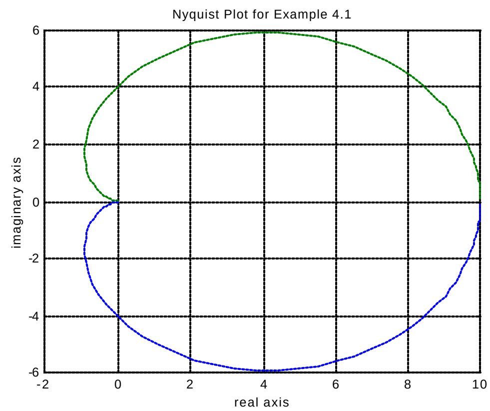


2. **Polar Plots and Nyquist Plots:** Polar ploats are the plots of *Im{H(jw)}* versus *Re{H(jw)}* with $s=jw_k$ as a parameter which varies from very small values to infinity. On the other hand, a Nyquist plot is a two-sided polar plot where the frequency is varied from $-\infty < w < \infty$.

Example 4.2: Obtain the polar and Nyquist plots for the previous Example 4.1.

```
%Example 4.2 Polar and Nyquist Plots for Example 4.1
num=[40]; den=[1,5,4];
w=logspace(-2,2,100);      %100 logarithmically spaced points
[re,im]=nyquist(num,den,w);   % call nyquist package
axis([-1,1,-1,1]); plot(re,im,'b')
title('Polar Plot for Example 4.1'); xlabel('real axis');
ylabel('imaginary axis');grid; axis; pause;
[re,im,w]=nyquist(num,den,w); axis([-1,1,-1,1]);
plot(re,im,re,-im); title('Nyquist Plot for Example 4.1')
xlabel('real axis'); ylabel('imaginary axis'); grid; axis;
```

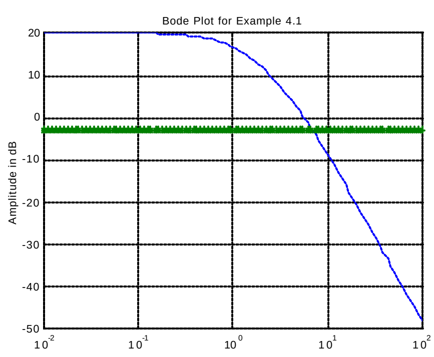


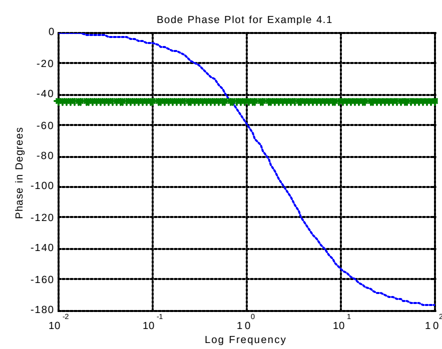


3. Bode Plots: These are semi-logarithmic plots of $20\log_{10}|H(jw)|$ *versus* w and P vs w.

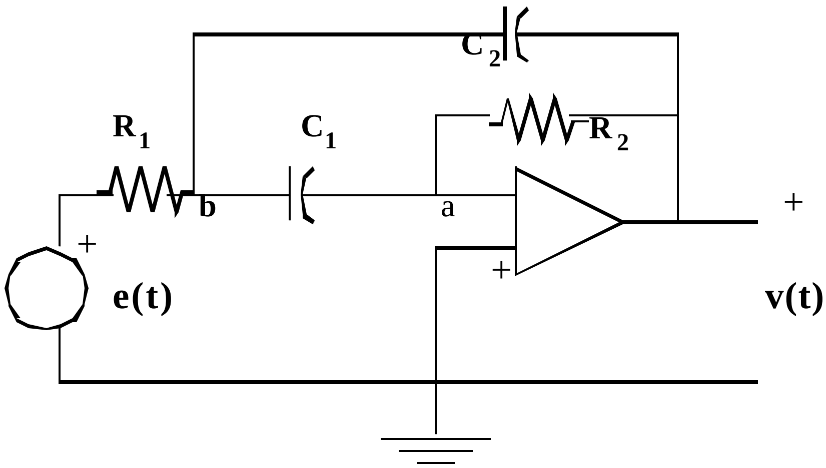
Example 4.3: Obtain the Bode plots for Example 4.1.

```
%Example 4.3 Bode Plots for Example 4.1
num=[40]; den=[1,5,4]; w=logspace(-2,2,100);    %100 logarithmic points
[amp,phase,w]=bode(num,den,w);   %call Bode package
amp=20*log10(amp); threedB=(max(amp)-3.0)*ones(size(w)); %-3.0 dB Reference
axis([-2,1,-60,10]); semilogx(w,amp,w,threedB,'*')
title('Bode Plot for Example 4.1'); xlabel('Log Frequency');
ylabel('Amplitude in dB'); grid; axis; pause;
D45=-45.0*ones(size(w));   %45 Degree reference Line
semilogx(w,phase,w,D45,'*'); title('Bode Phase Plot for Example 4.1')
xlabel('Log Frequency'); ylabel('Phase in Degrees'); grid; axis;
```





Example 4.4: Consider the following active OPAMP circuit, which is commonly know as a *second-order analog bandpass(BP) filter* in the field.



General transfer function (structure) of second-order BP circuits is of the form:

$$H_{BP}(s) = K\frac{s}{s^2 + Bs + w_0^2} = K\frac{s}{s^2 + \frac{w_0}{Q}s + w_0^2} \qquad (4.9)$$

where K is the D.C. gain, B is the 3.0 dB bandwidth, w_0 is the center or resonance frequency, and Q represents the quality factor at w_0 as a sharpness indicator of the resonance curve.

Let us analyze this circuit using Kirchoff's nodal (KCL) equations. KCL at point **b**:

$$V_b(s).[s(C_1 + C_2) + \frac{1}{R_1}] - sC_2.V(s) = \frac{E(s)}{R_1} \qquad (4.10)$$

Similarly, the KCL equation at point **a**:

$$sC_1.V(s) + V(s).\frac{1}{R_2} = 0 \qquad (4.11)$$

We can combine these two equations in a matrix form, which conforms well with Matlab analysis:

$$\begin{bmatrix} \frac{1}{R_1} + s(C_1 + C_2) & -sC_2 \\ sC_1 & \frac{1}{R_2} \end{bmatrix} . \begin{bmatrix} V_b(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} \frac{E(s)}{R_1} \\ 0 \end{bmatrix} \qquad (4.12)$$

Let us solve this linear system of equations for a specific case: $C_1 = C_2 = 1.0\ F$; $R_1 = \frac{1}{2Q}$; and $R_2 = 2Q$. When we substitute these into above we get:

$$\begin{bmatrix} 2Q+2s & -s \\ s & 2Q \end{bmatrix} . \begin{bmatrix} V_b(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} 2Q.E(s) \\ 0 \end{bmatrix} \tag{4.13}$$

results in:

$$V_b(s) = E(s).\frac{1}{s^2 + \frac{1}{Q}s + 1} \quad and \quad V(s) = E(s).\frac{-2Qs}{s^2 + \frac{1}{Q}s + 1} \tag{4.14}$$

rewriting these two equations yields the transfer function of this circuit:

$$H(s) \equiv \frac{V(s)}{E(s)} = \frac{-2Qs}{s^2 + \frac{1}{Q}s + 1} \quad \Leftrightarrow \quad H_{BP}(s) = \frac{s}{s^2 + \frac{w_0}{Q}s + w_0^2} \tag{4.15}$$

Our final result is identical to the general BP circuit structure with a resonance frequency $w_0 = 1.0 \ rad/s$ and a bandwidth: $B = \frac{1}{Q} = \frac{w_0}{Q} \Rightarrow B = \frac{1}{Q}$. The frequency response is obtained in the usual manner by the substitution:

$$H(jw) = H(s)\Big|_{s=jw} = \frac{-2jwQ}{-w^2 + \frac{jw}{Q} + 1} \quad \Rightarrow \quad H(j1) = \frac{-2jQ}{-1 + \frac{j}{Q} + 1} = -2Q^2 \tag{4.16}$$

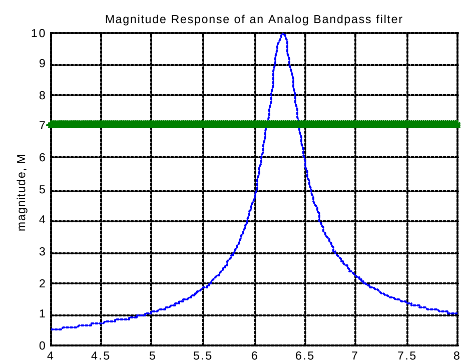
Let us note that the peak value of the frequency response will be simply $2Q^2$. If the center frequency is 10 KHz, the bandwidth is 500 Hz and $\left|H(jw)\right|_{max} = K/B = 10$, corresponding to $w_0 = 2p.10^4 \ rads/s$ and $B = 1000p \ rads/s$. We have $K = 10B = p.10^4$ to result in the final transfer function:

$$H_{BP}(s) = \frac{V(s)}{E(s)} = \frac{p.10^4.s}{s^2 + p.10^3.s + 4p^2.10^8} \tag{4.17}$$

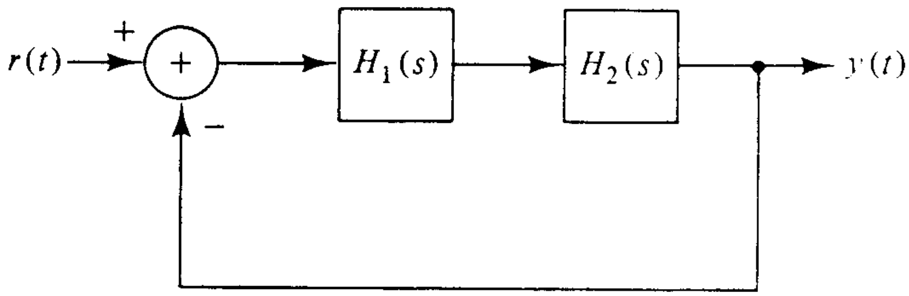
Let us perform the Matlab analysis to see the details of this result.

```
%Example 4.4 Magnitude Response of an Analog Bandpass filter
b=[31.415e3,0]; a=[1,3.145e3,39.478e8];
w=4e4:100:8e4;
H=freqs(b,a,w);   %Call Freq. Resp.
mag=abs(H);
g=0.707*ones(size(w))*abs(max(H));

axis([4e4,8e4,0,12]);
plot(w,mag,w,g,'*');
title('Magnitude Response');
xlabel('frequency, rad/s');
ylabel('magnitude, M');
grid; axis;
```



Example 4.5: Obtain the unit-step response of the following automatic position control system that is used in a tracking antenna (Example 5.8.4 from Soliman & Srinath).



Given that the first subsystem is a perfect amplifier and the second sub-system is a motor with transfer function:

$$H_1(s) = 8 \quad and \quad H_2(s) = \frac{1}{s(s+a)} \quad where \quad 0 < a < \sqrt{32}$$

The overall transfer function of this negative feedback configuration is simply:

$$H(s) = \frac{H_1(s).H_2(s)}{1 + H_1(s).H_2(s)} = \frac{8/s(s+a)}{1 + 8/s(s+a)} = \frac{8}{s^2 + a.s + 8} \tag{4.18}$$

The unit-step response is then the product of *H(s)* with *1/s*:

$$Y(s) = \frac{1}{s}.\frac{8}{s^2 + a.s + 8} = \frac{8}{s(s^2 + a.s + 8)} = \frac{1}{s} - \frac{s+a}{s^2 + a.s + 8} \tag{4.19}$$

The last result is obtained from PFE as previously discussed. It should be noted that the range $0 < a < \sqrt{32}$ is chosen to guarantee that the roots of the polynomial in (4.19) are in LHP to avoid unstability of the system. If we perform the inverse Laplace transform of *Y(s)* we get:

$$y(t) = \{1 - e^{-a.t/2}.Cos(t.\sqrt{8 - \frac{a^2}{4}}) + \frac{a}{2\sqrt{8 - \frac{a^2}{4}}} Sin(t.\sqrt{8 - \frac{a^2}{4}})\}.u(t) \tag{4.20}$$

We will next plot this step-response for a few values of a using Matlab. Let us try the following four values 0.1, 2, 4, and $\sqrt{32}$, which covers from the lower end of the allowed values all the way to the upper end.

```
% Example 4.5 Automatic Position-Control
% We will try alpha for 0.1, 2, 4, and sqrt(32).
num=[8]; den1=[1, 0.1, 8];
t=0:0.1:10;
y1=step(num,den1,t);subplot(221),plot(t,y1);
title('Step Response');xlabel('Time t');ylabel('Output, y(t)');
den2=[1, 2, 8];
y2=step(num,den2,t);subplot(222),plot(t,y2);
den3=[1, 4, 8];
y3=step(num,den3,t);subplot(223),plot(t,y3);
den4=[1,sqrt(32),8];
y4=step(num,den4,t);subplot(224),plot(t,y4);
```