Chapter on Laplace Transforms and Applications

In classical systems analysis and design, the s-domain treatment of signals and system has played a significant role. With the emergence of digital signal processing and the wide range usage of computer-based analysis and design, this role has been significantly reduced and it has been greatly replaced by discrete (or more appropriately, fast) Fourier transform and z-transform techniques. Nevertheless, basic treatment of Laplace transform forms a solid bridge from classical circuits and systems to modern systems.

For causal --physically realizable-- systems the *one-sided Laplace Transform* is defined by:

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t).e^{-st}\,dt \quad where\ s = s + jw \tag{3.1}$$

where s is a complex variable is s-plane whose real-axis is designated with s and the imaginary axis with jw. The region of s-plane for which the above integral is convergent is called the *Region of Convergence (ROC)*. Similarly, the *inverse Laplace Trans*form is defined by:

$$x(t) = L^{-1}\{X(s)\} = \int_{s - j\infty}^{s + j\infty} X(s).e^{st}\,ds \tag{3.2}$$

These two equations are normally represented in the form of a pair relationship:

$$x(t) \Leftrightarrow X(s) \tag{3.3}$$

3.1 Basic Laplace Transform Pairs

1. Impulse (Delta) Function:

$$L\{d(t)\} = \int_0^{\infty} d(t).e^{-st}\,dt = e^{-0} = 1 \tag{3.4}$$

This result is from the Sifting Theorem definition of a delta function. Similarly, we can compute the Laplace transform of a generic delta function using the same theorem:

$$L\{d(t - t_0)\} = \int_0^{\infty} d(t - t_0).e^{-st}\,dt = e^{-st_0} \quad if\ t_0 > 0 \tag{3.5}$$

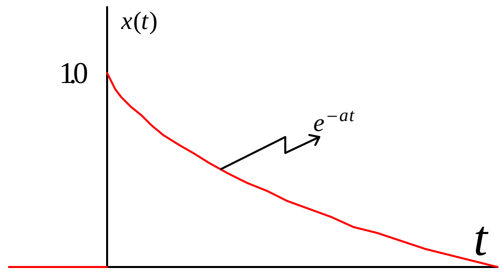
Thus, we can conclude that:

$$d(t) \Leftrightarrow 1 \qquad and \qquad d(t - t_0) \Leftrightarrow e^{-st_0} \tag{3.6}$$

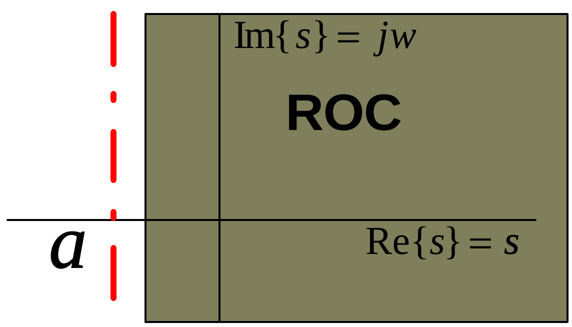
2. Step Function:

$$L\{u(t)\} = \int_0^\infty u(t).e^{-st}dt = \int_0^\infty 1.e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s} \qquad (3.7)$$

3. Real Exponential Causal Function:



$$L\{x(t)\} = L\{e^{-at}.u(t)\} = \int_0^\infty e^{-at}.u(t).e^{-st}dt = \int_0^\infty e^{-(a+s)t}.e^{-st}dt = \left.\frac{e^{-(a+s)t}}{-(a+s)}\right|_0^\infty = \frac{1}{s+a} \qquad (3.8)$$



4. **Ramp Signal:**

$$x(t) = t.u(t) \qquad (3.9)$$

$$L\{x(t)\} = L\{t.u(t)\} = \int_0^\infty t.u(t).e^{-st}dt = \frac{1}{s^2} \qquad (3.10)$$

This last result is easily obtained using the integration-by-parts rule in calculus.

3.2 Properties of Laplace Transforms

1. Linearity:

$$a.x_1(t) + b.x_2(t) \Leftrightarrow a.X_1(s) + b.X_2(s) \qquad (3.11)$$

Example 3.1: Consider a second-order system described in s-domain rather than the time-domain, which has be the case until now:

$$H(s)=\frac{1}{(s+a).(s+b)} \quad where \quad a\neq b \tag{3.12}$$

Find the system impulse response

$$H(s)=\frac{1}{(s+a).(s+b)}=\frac{C}{s+a}+\frac{D}{s+b}=C.V_1(s)+D.V_2(s) \tag{3.13}$$

Let us use partial fraction expansion to obtain C and D:

$$\frac{C(s+b)+D(s+a)}{(s+a)(s+b)}=\frac{1}{(s+a)(s+b)} \quad \Rightarrow C(s+b)+D(s+a)=1+0.s$$

These two unknowns can be obtained by equating equal powers of s on both sides:

$$C+D=0 \quad and \quad Cb+Da=1 \quad \Rightarrow D=-C=\frac{1}{a-b} \tag{3.14}$$

Substitution of these into (3.13) gives:

$$H(s)=\frac{1/(b-a)}{s+a}+\frac{1/(a-b)}{s+b} \tag{3.15}$$

Using the result in (3.8) we get the inverse Laplace transform term-by-term:

$$h(t)=\frac{1}{b-a}.e^{-at}u(t)+\frac{1}{a-b}.e^{-bt}u(t) \tag{3.16}$$

It should be obvious now why we wanted to force $a\neq b$.

2. Time-Differentials:

$$\frac{d}{dt}x(t)\Leftrightarrow s.X(s)-x(0) \tag{3.17}$$

where $x(0)$ stands for the value of *x(t)* at time *t=0*. Similarly, the higher-order differentials can be written:

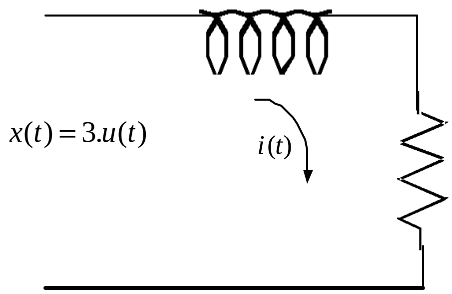
$$\frac{d^N}{dt^N}x(t)\Leftrightarrow s^N.X(s)-s^{N-1}.x(t)\Big|_{t=0}-s^{N-2}.\frac{d}{dt}x(t)\Big|_{t=0}-\cdots-\frac{d^{N-1}}{dt^{N-1}}x(t)\Big|_{t=0} \tag{3.18}$$

Example 3.2: Consider a second-order derivative case:

$$\frac{d^2}{dt^2}x(t)\Leftrightarrow s^2.X(s)-s.x(0)-\frac{d}{dt}x(t)\Big|_{t=0}=s^2.X(s)-s.x(0)-x'(0) \tag{3.19}$$

Example 3.3: If $i(0^-) = 5.0\ Amps$ find the current flowing in the loop of this circuit.

Solution Stage:



$L=2H\ \ and\ R=4\Omega$

From circuit theory we write the Kirchoff's Voltage Law for this loop and term-by-term transform:

$$3.u(t) = 2.\frac{d}{dt}i(t) + 4.i(t) \quad \Leftrightarrow \quad 3.\frac{1}{s} = 2[s.I(s) - i(0)] + 4.I(s) \qquad (3.20)$$

$$(2s+4).I(s) = \frac{3}{s} + 2.i(0) = \frac{3}{s} + 10$$

$$I(s) = \frac{\frac{3}{s}+10}{2(s+2)} = \frac{1.5+5s}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \qquad (3.21)$$

As we have done in a previous example, performing partial fraction expansion results in:

$$A = 0.75 \quad and \quad B = 4.25$$

Substituting these into (3.21) and taking term-by-term inverse Laplace transform we have the desired result:

$$I(s) = \frac{0.75}{s} + \frac{4.25}{s+2} \quad \Leftrightarrow \quad i(t) = 0.75u(t) + 4.25e^{-2t}.u(t) \qquad (3.22)$$

Verification Stage:

Our result can be accepted ONLY IF they are verified.

a. Let us substitute *t=0* to check the IC condition: $i(0) = 0.75 + 4.25 = 5.0\ Amps.$

b. The final value: As $t \to \infty$, the current in L should be totally discharged and we have 3.0 Volts across a 4 Ohms resistor to result in $i_{ss}(t) = 0.75\ Amps$. Now let us apply the same limiting operation to our answer in (3.22):

$$i_{t\to\infty}(t) = 0.75u(t)\big|_{t\to\infty} + 4.25e^{-2t}.u(t)\big|_{t\to\infty} = 0.75 + 0 = 0.75\ Amps. \qquad (3.23)$$

as expected and thus we have verified our solution be checking its validity at two critical values.

3. Time-Integrals:

$$\int_0^t x(t)dt \Leftrightarrow \frac{X(s)}{s} \qquad (3.24)$$

Example 3.4: Using the above integration property find: $L\{\sin(wt).u(t)\}$

It is worth noting that we could attempted to solve this problem by direct definition of Laplace transform of (3.1), the solution of which would have required somewhat advanced table of integrals. Alternatively, we could use Euler's Relation from Trigonometry to avoid that:

$$Sin(wt) = \frac{1}{2j}e^{jw} - \frac{1}{2j}e^{-jw}$$

or for our case at hand:

$$Sin(wt).u(t) = (\frac{1}{2j}e^{jw} - \frac{1}{2j}e^{-jw}).u(t) \qquad (3.25)$$

Let us take Laplace transform of both sides term-by-term:

$$L\{Sin(wt).u(t)\} = \frac{1}{2j}\frac{1}{s-jw} - \frac{1}{2j}\frac{1}{s+jw} = \frac{s+jw-s+jw}{2j(s+jw)(s-jw)} = \frac{2jw}{2jw(s^2+w^2)} = \frac{w}{s^2+w^2} \qquad (3.26)$$

Note that we could similarly evaluate and find that:

$$L\{Cos(wt).u(t)\} = \frac{s}{s^2+w^2} \qquad (3.27)$$

4. Time-Delays and Frequency-Shifts:

$$x(t-a).u(t-a) \Leftrightarrow e^{-as}.X(s) \quad if\ a>0 \qquad (3.28)$$

and

$$X(s-c) \Leftrightarrow e^{at}x(t) \qquad (3.29)$$

First of these properties forms the basis for digital signal processing, whereas the second one will reappear in the form of *Modulation Theorem* to lay the ground work for communication systems.

5. Convolution Property:

From convolution theorem of the previous chapter, we know that

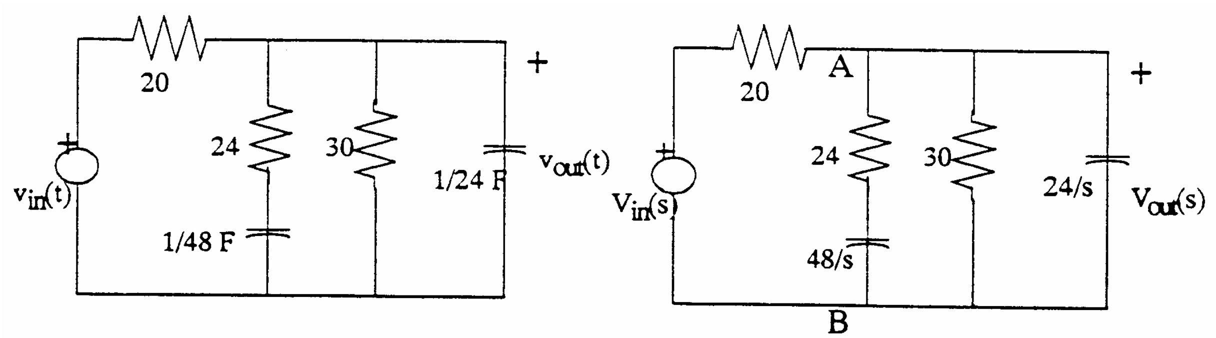
$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-t)dt = \int_{-\infty}^{\infty} h(t)x(t-t)dt \qquad (3.30)$$

The equivalent of this integral, which is not very easy to perform in many cases, is simply a multiplication in s-domain:

$$Y(s) = X(s).H(s) = H(s).X(s) \qquad (3.31)$$

Because of its simplicity almost all the time we replace the convolution operation with a multiplication process after appropriately taking the Laplace transforms involved.

Example 3.5: Let us illustrate the importance of this equivalent in the following circuit theory problem. Given that the *RC-network* has no energy stored in the circuit at *t=0*, (a) find the impulse response *h(t)* the output voltage for (b) $v_{in}(t) = d(t)$ and (c) $v_{in}(t) = 50Cost(2t).u(t)\ Volts$.



a. Since there are two capacitors in the network we should expect a second order ODE, which is fairly tedious to solve in t-domain, instead let us re-draw the circuit is s-domain. There are number of circuit theory techniques to solve this problem, which include Kirchoff's Laws, mesh and nodal analysis, Thevenin equivalent circuits, and source transformations. Here we will use the impedance-cut method. Let us define the impedance or the admittance to the right of AB-cut in the circuit as $Z_{in}(s) \equiv 1/Y_{in}(s)$:

$$Y_{in}(s) = \frac{s}{24} + \frac{1}{30} + \frac{1}{24 + \frac{48}{s}} = \frac{5s^2 + 19s + 8}{120(s+2)} \tag{3.32}$$

$$Z_{in}(s) = \frac{120(s+2)}{5s^2 + 19s + 8} \tag{3.33}$$

Next we apply voltage division rule to the reduced circuit:

$$V_{out}(s) = V_{in}(s)\frac{Z_{in}(s)}{20 + Z_{in}(s)} \tag{3.34}$$

Let us define the system transfer function as the output to input voltage ration in s-domain:

$$H(s) \equiv \frac{V_{out}(s)}{V_{in}(s)} = \frac{Z_{in}(s)}{20 + Z_{in}(s)} = \frac{\frac{120}{5s^2 + 19s + 8}}{20 + \frac{120}{5s^2 + 19s + 8}} = \frac{6(s+2)}{5s^2 + 25s + 20}$$

$$= \frac{1.2(s+2)}{s^2 + 5s + 4} = \frac{A}{s+1} + \frac{B}{s+4}$$

Solution of the partial fraction expansion in the last equality yields: *A=0.4* and *B=0.8.*

$$H(s) = \frac{A}{s+1} + \frac{B}{s+4} = \frac{0.4}{s+1} + \frac{0.8}{s+4} \tag{3.35}$$

The next step is to find the impulse response *h(t)* via Laplace transform of (3.35):

$$h(t) = 0.4e^{-t}u(t) + 0.8e^{-4t}u(t) = (0.4e^{-t} + 0.8e^{-4t}).u(t) \quad (3.36)$$

b. Let us now obtain the output for delta input:

$$v_{in}(t) = d(t)$$

$$V_{out}(t) = V_{in}(t) * h(t) = d(t) * (0.4e^{-t} + 0.8e^{-4t}).u(t) = (0.4e^{-t} + 0.8e^{-4t}).u(t) \quad (3.37)$$

c. Similarly, the output for a sinusoidal input $v_{in}(t) = 50Cos(2t).u(t)$ would require convolution of a sinusoidal function with a pair of exponential functions, which is rather difficult to evaluate. Instead let us replace the convolution with a multiplication in s-domain:

$$V_{in}(s) = 50\frac{s}{s^2+4}$$

$$V_{out}(s) = 50\frac{s}{s^2+4}.H(s) = 50\frac{s}{s^2+4}.\frac{1.2(s+2)}{s^2+5s+4} = \frac{60s(S+2)}{(s+1)(s+4)(s^2+4)} \quad (3.38)$$

This third-order system can be partial fractioned into a number of smaller order terms:

$$V_{out}(s) = \frac{-4}{s+1} + \frac{-8}{s+4} + \frac{12s}{s^2+4} + \frac{2}{s^2+4} \quad (3.39)$$

(The details of this type of partial fraction expansion will be discussed later.)

$$V_{out}(t) = \{-4e^{-t} - 8e^{-4t} + 12Cos(2t) + 2Sin(2t)\}.u(t) \quad (3.40)$$

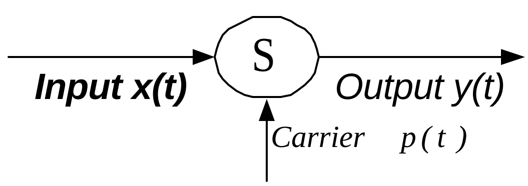
If we carefully study (3.40) we see that the first two terms represent the transient response, which converges to zero since the system was defined to be at rest at *t=0*. The last two terms stand for the steady-state response and they oscillate forever with a frequency of *w=2 rad/s.*

6. S-Domain Properties:

Shifts or translation: If $x(t) \Leftrightarrow X(s) \quad then \quad X(s+c) \Leftrightarrow e^{-ct}.x(t)$ (3.41)

Scaling: If $x(t) \Leftrightarrow X(s) \quad then \quad \frac{1}{k}X(\frac{s}{k}) \Leftrightarrow x(kt)$ (3.42)

Convolution in S-Domain (Mixing): This is simply the dual of the time-convolution and it is commonly known as mixing in electronics or modulation in communications engineering world:



$$y(t) = x(t).p(t) \quad and \quad Y(s) = \int_{-\infty}^{\infty} X(x).P(s-x)dx \quad (3.43)$$

7. Initial Value Theorem: $Lim_{t\to 0}\, x(t) = Lim_{s\to\infty}\,[s.X(s)]$ (3.44)

8. Final Value Theorem: $Lim_{t\to\infty} x(t) = Lim_{s\to 0}[s.X(s)]$ (3.45)

3.3 Canonical Implementations Using Laplace Transforms

1. Transfer Function Representation:

$$H(s) \equiv \frac{Y(s)}{X(s)} \tag{3.46}$$

2. Differential Equation Representation in s-Domain:

Consider the ODE of (2.34) and more appropriately the operator form that in (2.36) with the usual assumption of $M=N=Order$:

$$\frac{d^N}{dt^N} y(t) = \sum_{i=0}^{M} b_i \frac{d^i}{dt^i} x(t) - \sum_{j=0}^{N-1} a_j \frac{dj}{dt^j} y(t)$$

$$D^N y(t) = \sum_{i=0}^{N} b_i D^i x(t) - \sum_{j=0}^{N-1} a_j D^j y(t)$$

Assuming all initial conditions are zero and $a_N \equiv 1$, replacing D with s in the last expression yields the corresponding representation in s-domain:

$$s^N Y(s) = \sum_{i=0}^{N} b_i s^i X(s) - \sum_{j=0}^{N-1} a_j s^j Y(s)$$

$$s^N Y(s) + \sum_{j=0}^{N-1} a_j s^j Y(s) = \sum_{i=0}^{N} b_i s^i X(s) \tag{3.47}$$

$$\sum_{j=0}^{N} a_j s^j Y(s) = \sum_{i=0}^{N} b_i s^i X(s)$$

3. Rational Transfer Function Representation in s-Domain:

If we form (3.46) from the last equality in (3.47) and continue to assume all IC's are zero we get:

$$H(s) \equiv \frac{N(s)}{D(s)} = \frac{Y(s)}{X(s)} = \frac{\sum_{i=0}^{N} b_i s^i}{\sum_{j=0}^{N} a_j s^j} = \frac{\sum_{i=0}^{N} b_i s^i}{a_N s^N + \sum_{j=0}^{N-1} a_j s^j} \tag{3.48}$$

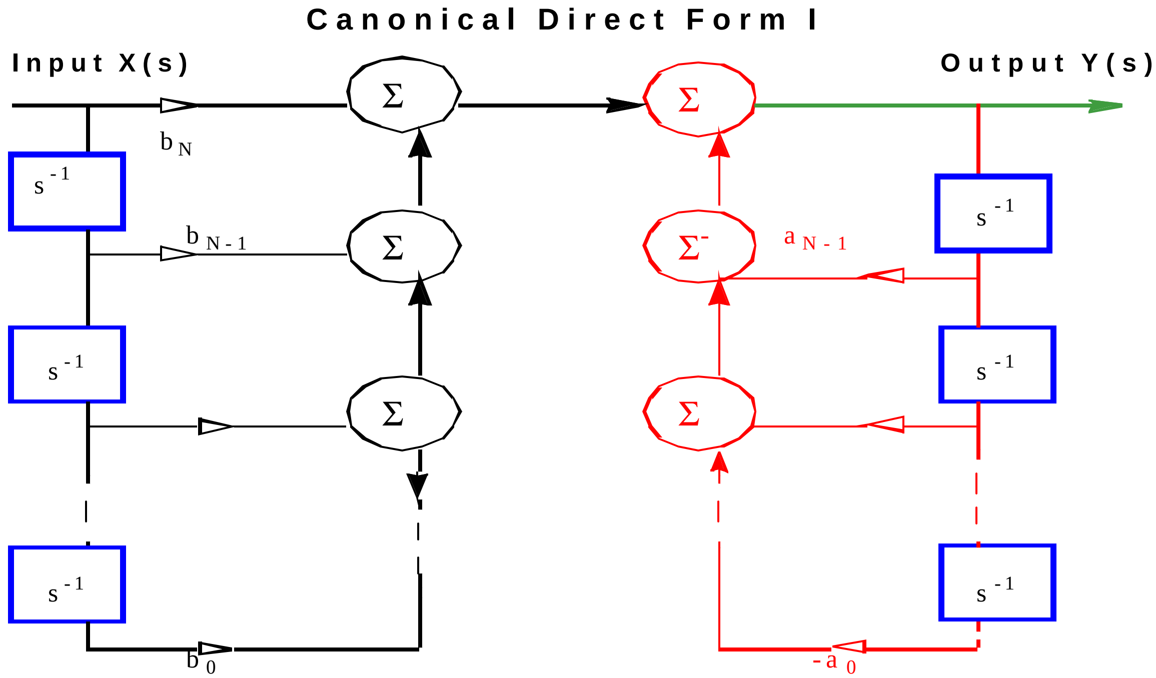
Here we have a numerator polynomial *N(s)*, solution of that gives the location of zeros of the system, and a denominator polynomial *D(s)*, which has the information on poles of the system.

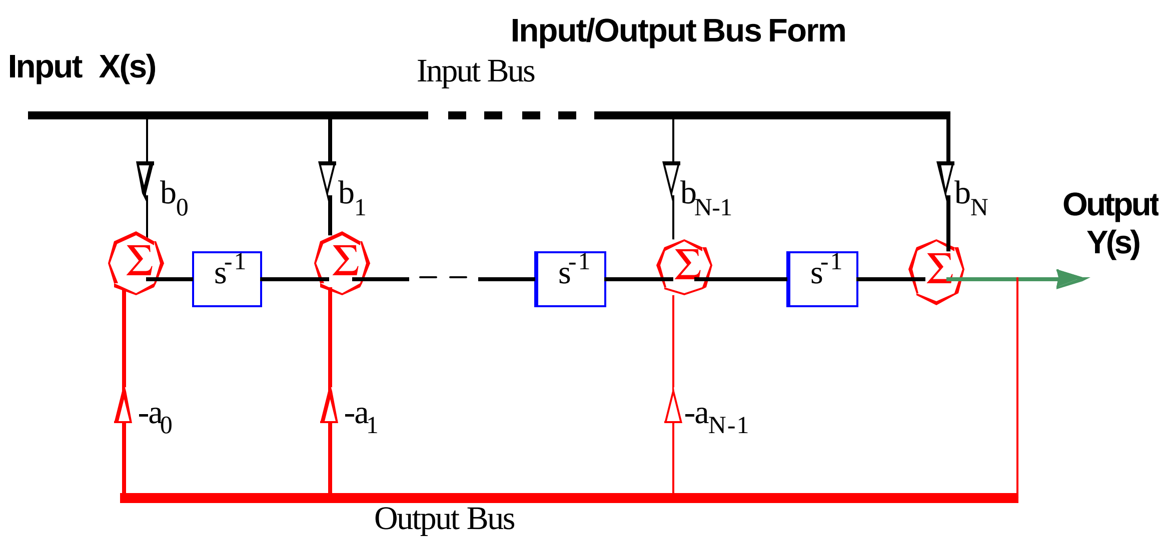
4. Canonical Implementation Diagrams in s-Domain:

If we recall from previous chapter, the last form of ODE we have employed for implementation was (2.38). Let us revisit that in s-domain:

$$\begin{aligned} Y(s) &= b_N X(s) + \frac{1}{s^1}[b_{N-1}X(s) - a_{N-1}Y(s)] + \frac{1}{s^2}[b_{N-2}X(s) - a_{N-2}Y(s)] \\ &+ \cdots + \frac{1}{s^{N-1}}[b_1 X(s) - a_1 Y(s)] + \frac{1}{s^N}[b_0 X(s) - a_0 Y(s)] \end{aligned} \tag{3.49}$$

As in the previous case we have Canonical Direct Form I and I/O-Bus Structure implementation options We have canonical (standard) implementation forms based on simple building blocks of a delay, adder, an scalar multiplier.





Example 3.6: Given a third-order system with a transfer function:

$$H(s) = \frac{s^2 + 100}{s^3 + 20s^2 + 200s + 1000}$$

It is easy to see that $N=3$ and $M=2$, i.e., there are two zeros and three poles of this system.

$$Y(s).(s^3+20s^2+200s+1000)=X(s).(s^2+100)$$

$$\frac{Y(s)}{s^3}(s^3+20s^2+200s+1000)=\frac{X(s)}{s^3}(s^2+100)$$

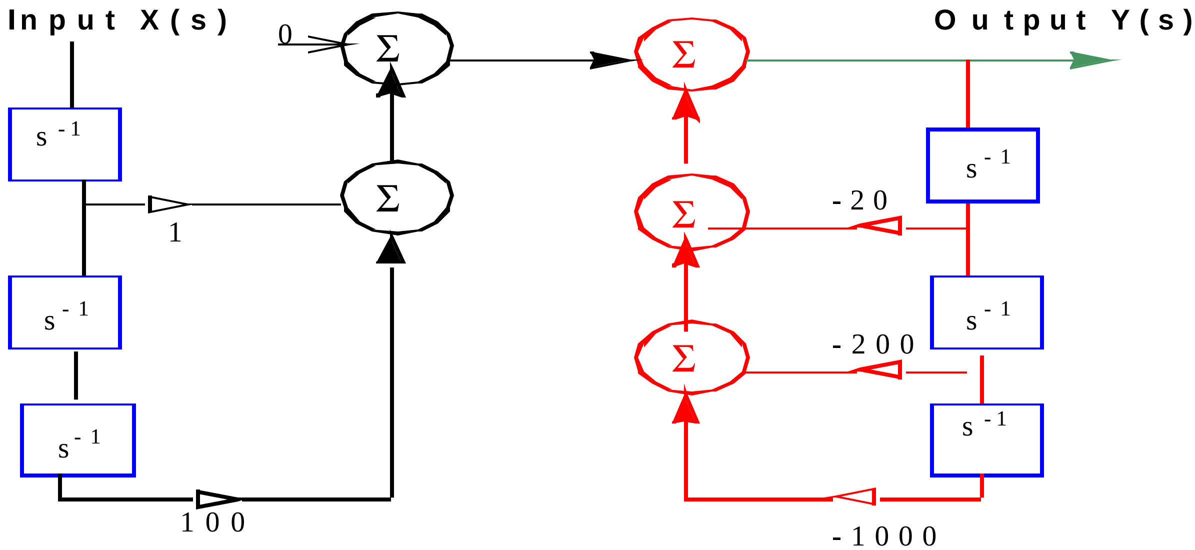
$$Y(s).(1+20s^{-1}+200s^{-2}+1000s^{-3})=X(s).(s^{-1}+100s^{-3})$$

$$Y(s)=X(s).(s^{-1}+100s^{-3})-(20s^{-1}+200s^{-2}+1000s^{-3})Y(s)$$

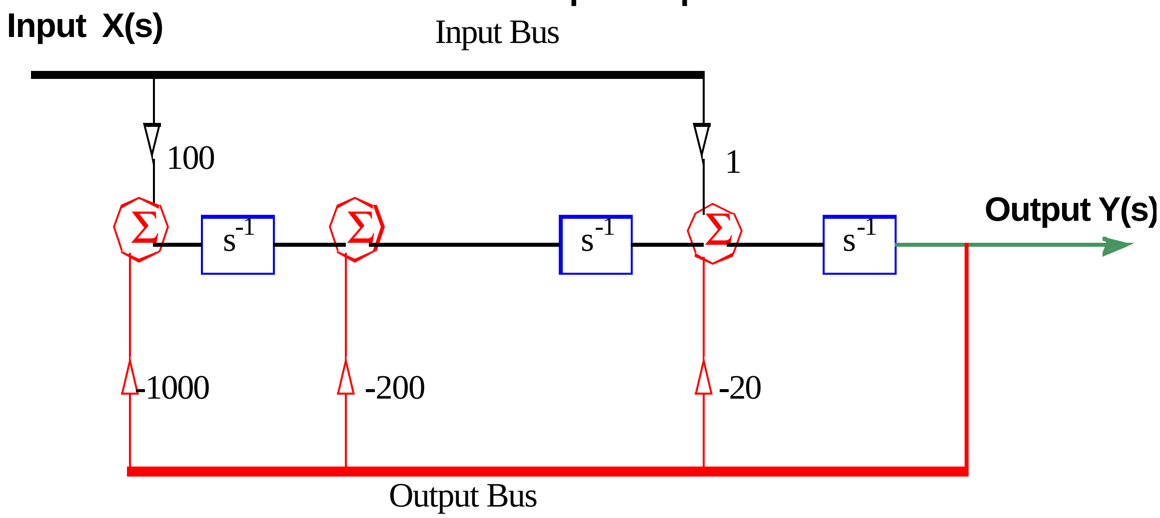
The direct implementation is on left below and the form for I/O-Bus implementation on the right:

$$Y(s)=s^{-1}[1.X(s)-20.Y(s)]+s^{-2}[0X(s)-200Y(s)]+s^{-3}[100X(s)-1000Y(s)]$$

Canonical Form I



Input/Output Bus Form



3.4 Partial Fraction Expansion and Examples

1. Simple Poles:

$$H(s) \equiv \frac{Y(s)}{X(s)} = \frac{N(s)}{D(s)} = K.\frac{(s-z_1)(s-z_2)\cdots(s-z_L)}{(s-p_1)(s-p_2)\cdots(s-p_N)} \qquad (3.50)$$

- K= Gain of the system
- Normally $L<N;$ with zeros at locations: $s_1=z_1$; $s_2=z_2$; …; $s_L=z_L$.
- In addition, there are N-L zeros at infinity: $s_i = \pm\infty$.
- There are N simple poles (poles of power one) at locations: $s_1=p_1$; $s_2=p_2$; …; $s_N=p_N$.

Since all the poles are simple poles, we can expand *H(s)* into a rational form by long-division:

$$H(s) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_N}{s-p_N} \qquad (3.51)$$

The impulse response of H(s) is very simple to find for these first-order cases, i.e.:

$$h(t) = \sum_{k=1}^{N} C_k . e^{p_k t} . u(t) \qquad (3.52)$$

To tackle problems of this type, we can utilize ***[r,p,k]=residue(b,a)*** command in Matlab. The notation is as follows:

- **b:** Numerator coefficients matrix of *H(s)* in descending order of *s*.
- **a:** Denominator coefficients matrix of *H(s)* in descending order of *s*.
- $\mathbf{r} = \{C_i\}$; constant coefficients in (3.51).
- $\mathbf{p} = \{p_i\}$; pole locations in (3.51).
- $k = \{0\}$; Irrational portion of *H(s)*. For rational system functions this matrix is zero as expected. When $L=N$, that will require $k=1$, and when $L>N$ then there will be an appropriate set of non-zero values for *k-matrix* depending upon the specific case.

Example 3.7: Let us find the poles of the following transfer function:

$$H(s) = \frac{1}{s^2 - 3.5s + 1.5}$$

$$b = [1]; \quad a = [1, -3.5, 1.5]; \quad [r, p, k] = residue(b, a);$$

$$\mathbf{r} = \begin{bmatrix} 0.4 \\ -0.4 \end{bmatrix}; \quad \mathbf{p} = \begin{bmatrix} 3 \\ 0.5 \end{bmatrix}; \quad \mathbf{k} = [\ ]$$

This result will permit us to write the transfer function in a rational form:

$$H(s) = \frac{0.4}{s-3} - \frac{0.4}{s-0.5}$$

2. Simple Poles with $L=N$:

$$H(s) \equiv \frac{Y(s)}{X(s)} = \frac{N(s)}{D(s)}$$

We long-divide N(s) by D(s) to transform H(s) a rational (proper) function plus a constant C_0.

$$H(s) = C_0 + \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_N}{s-p_N} = C_0 + \sum_{k=1}^{N} \frac{C_k}{s-p_k} \qquad (3.53)$$

The impulse response for this case is very similar to the previous case, except a delta function to include the effect of improper (irrational) term C_0:

$$h(t) = C_0 . d(t) + \sum_{k=1}^{N} C_k . e^{p_k t} . u(t) \qquad (3.54)$$

Example 3.8: Let us find the poles of the following transfer function:

$$H(s) = \frac{s^2-9}{s^2-1} = 1 - \frac{8}{s^2-1} = 1 - \frac{8}{(s+1)(s-1)} = 1 + \frac{4}{s+1} - \frac{4}{s-1}$$

$$b = [1;0;-9]; \quad a = [1,0,-1]; \quad [r,p,k] = residue(b,a);$$

$$\mathbf{r} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}; \quad \mathbf{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}; \quad \mathbf{k} = [1]$$

$$h(t) = d(t) + 4e^{-t}.u(t) - 4e^{t}.u(t) \quad ROC : Re\{s\} > 1.$$

3. Complex Poles of Order l :

The first step is to check if the transfer function *H(s)* is rational (proper). If it is improper we partition the system function into an improper part plus a rational part by long division as in the previous case. After that, we get the following form for the proper part:

$$H(s) = \frac{C_1}{s-p_1} + \frac{C_1^*}{s-p_1^*} + \frac{C_2}{s-p_2} + \frac{C_2^*}{s-p_2^*} + \cdots \qquad (3.55)$$

The impulse response in this case contains a pair of complex conjugate terms for each pair of poles in (3.55)

$$h(t) = C_1.e^{p_1 t}.u(t) + C_1^*.e^{p_1^* t}.u(t) + C_2.e^{p_1 t}.u(t) + C_2^*.e^{p_2^* t}.u(t) + \cdots \qquad (3.56)$$

Example 3.9: Let us find the poles of the following transfer function us ing Matlab analysis:

$$H(s) = \frac{8}{s^2+10s+169}$$

$$b = [8]; \quad a = [1,10,169]; \quad [r,p,k] = residue(b,a);$$

$$\mathbf{r} = \begin{bmatrix} -j/3 \\ j/3 \end{bmatrix}; \quad \mathbf{p} = \begin{bmatrix} -5+j12 \\ -5-j12 \end{bmatrix}; \quad \mathbf{k} = [\ ]$$

$$H(s) = \frac{-j/3}{s+5-j12} + \frac{j/3}{s+5+j12}$$

The impulse response will have a pair of complex conjugate terms:

$$h(t) = \frac{-j}{3} e^{-(s-j12)t}.u(t) + \frac{j}{3} e^{-(s+j12)t}.u(t).$$

1. **Multiple (Repeating) Poles:** We will study this case with two examples.

Example 3.10: For all *s*, such that: *Re{s}>0*:

$$H(s)=\frac{2s+12}{(s+1)^2}=\frac{C_1}{(s+1)^2}+\frac{C_2}{s+1}$$

$$C_1=H(s)(s+1)^2\Big|_{s=-1}=\frac{2s+12}{(s+1)^2}(s+1)^2\Big|_{s=-1}=2s+12\Big|_{s=-1}=10$$

$$C_2=H(s)(s+1)\Big|_{s=-1}=\frac{2s+12}{(s+1)^2}(s+1)\Big|_{s=-1}=[\frac{C_1}{(s+1)^2}(s+1)+\frac{C_2}{s+1}(s+1)]\Big|_{s=-1}$$

$$\frac{2s+12}{(s+1)^2}=\frac{10}{s+1}+C_2$$

$$C_2=[\frac{2s+12-10}{s+1}]\Big|_{s=-1}=2$$

$$H(s)=\frac{10}{(s+1)^2}+\frac{2}{s+1}$$

We can find the impulse response by using the particular entries in standard Laplace transform tables:

$$h(t)=10.t.e^{-t}.u(t)+2.e^{-t}.u(t).$$

Example 3.11:

$$H(s)=\frac{s^4+2s^3+3s^2+2s+1}{s^4+4s^3+7s^2+6s+2}$$

As in few examples the Matlab assisted solution for this fourth-order system is given by:

$b=[1,2,3,2,1];\quad a=[1,4,7,6,2];\quad [r,p,k]=residue(b,a);$

$$\mathbf{r}=\begin{bmatrix}-j0.5\\ j0.5\\ -2\\ 1\end{bmatrix};\quad \mathbf{p}=\begin{bmatrix}-1+j\\ -1-j\\ -1\\ -1\end{bmatrix};\quad \mathbf{k}=[1]$$

$$H(s)=1-\frac{j0.5}{s+j-1}+\frac{j0.5}{s+j+1}-\frac{2}{s+1}+\frac{1}{(s+1)^2}$$

The impulse response from Laplace Transform Tables will be:

$$h(t)=d(t)+e^{-t}.Cos(t-\frac{p}{2}).u(t)-2e^{-t}.u(t)+t.e^{-t}.u(t)$$

Example 3.12: Given the impulse response of a system find its *step-response*:

$$h(t)=(e^{-t}-e^{-2t}).u(t)\quad and\quad recall: x(t)=u(t)$$

a. Let us find the step-response without performing the convolution operation:

$$y(t)=h(t)*x(t)=h(t)*u(t)$$

$$X(s)=L\{u(t)\}=1/s\quad and\quad H(s)=\frac{1}{s+1}-\frac{1}{s+2}=\frac{1}{s^2+3s+2}$$

$$Y(s) = X(s).H(s) = \frac{1}{s}.\frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{s^3 + 3s^2 + 2s}$$

b. Using the values obtained via Matlab analysis and the Laplace transform tables we have:

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} [e^{-t} - e^{-2t}]u(t)u(t-t)dt = \int_0^t e^{-t}dt - \int_0^t e^{-2t}dt$$

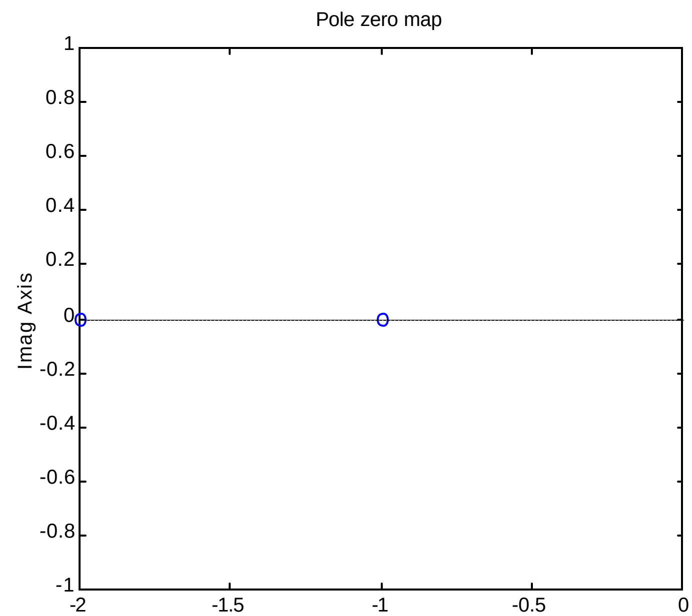
$$= (0.5 - e^{-t} + 0.5e^{-2t})u(t)$$

c. Using the values obtained via Matlab analysis and the Laplace transform tables we have:

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} [e^{-t} - e^{-2t}]u(t)u(t-t)dt = \int_0^t e^{-t}dt - \int_0^t e^{-2t}dt$$

$$= (0.5 - e^{-t} + 0.5e^{-2t})u(t)$$

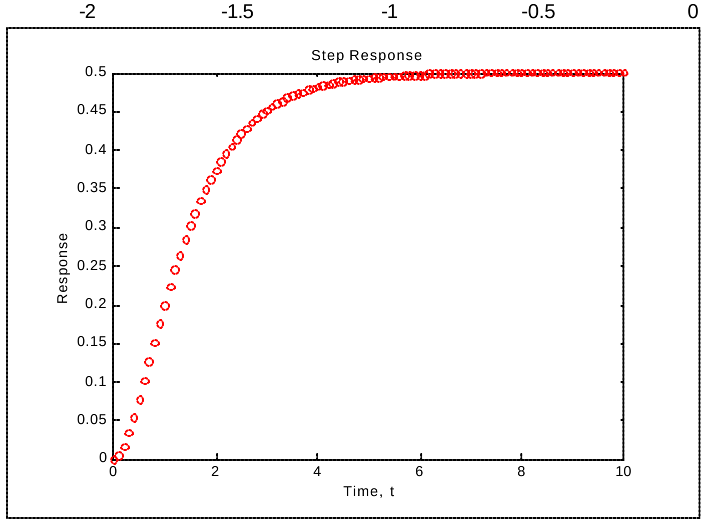
```
% ILLUSTRATION OF EXAMPLE
%Numerator & Denominator Sets
b=[1]; a=[1,3,2];

%Plot pole-zero map
pzmap(b,a); pause;

%Revise the arrays
% for step-response
newb=1; newa=[1,3,2,0];
[r,p,k]=residue(newb,newa)
pause;
```



r =0.5000   p =-2   k=[]
   -1.0000   -1
   0.5000   0

-1.0000
0.5000

```
%Step function verification
t=0:0.1:10;
y=step(b,a,t);
plot(t,y,'O');
xlabel('Time, t');
ylabel('Response');
title('Step Response');
axis
```



SUMMARY OF USEFUL PFE FORMS

Form	s-Domain	t-Domain
1	$\dfrac{K}{s+a}$	$K.e^{-at}.u(t)$
2	$\dfrac{K}{(s+a)^2}$	$K.t.e^{-at}.u(t)$
3	$\dfrac{K}{s+a-jb}+\dfrac{K^*}{s+a+jb}$	$2.\lvert K\rvert.e^{-at}.Cos(bt+q).u(t)$
4	$\dfrac{K}{(s+a-jb)^2}+\dfrac{K^*}{(s+a+jb)^2}$	$2.\lvert K\rvert.t.e^{-at}.Cos(bt+q).u(t)$

Notes:

- In (1,2) K is a real constant.
- In (3,4) K is a complex parameter with a form:
- $K = c + jd = \lvert K\rvert .e^{j.q} \quad where\ \lvert K\rvert = \sqrt{c^2+d^2} \quad and\ q = a\tan(\frac{d}{c})$

3.5 System Stability from Pole-Zero Locations

Case 1. Simple Poles with *L*<*N*:

$$H(s)=\frac{N(s)}{D(s)}=\frac{C_1}{s-p_1}+\frac{C_2}{s-p_2}+\cdots+\frac{C_N}{s-p_N}=\frac{C_1}{s-(s_1+jw_1)}+\cdots+\frac{C_N}{s-(s_N+jw_N)} \qquad (3.57)$$

Conclusions:

a. If all p_i are real then it is obvious that: $w_1 = w_2 = \cdots = w_N \equiv 0$ and impulse response is simply a sum of real exponentials:

$$h(t)=(C_1e^{s_1t}+C_2e^{s_2t}+\cdots+C_Ne^{s_Nt}).u(t) \qquad (3.58)$$

- If all $s_i < 0$ then the roots (poles) are on negative real axis and each exponential term decays as *t* increases. Thus, the system is *unconditionally stable* for bounded inputs.
- If any $s_i > 0$ then some of the poles are on the positive real-axis and as a result the term for this particular pole will diverge as t increases. The system will *be generally unstabl*e for any input. However, it is *unconditionally unstable for bounded inputs.*

b. If the system has complex roots then they must occur in complex conjugate pairs, i.e., if $s_1 + jw_1$ is a root with a coefficient C_1, then $s_1 - jw_1$ is also a root with constant C_1^*. For a simple pole-pair at location: $s_k = s_k \pm jw_k$ the impulse response due to these two terms will be:

$$h_k(t) = C_k.e^{s_k t}.e^{jw_k t}.u(t) + C_k^*.e^{s_k t}.e^{-jw_k t}.u(t)$$
$$= 2|C_k|.e^{s_k t}.Cos(w_k t + q_k) \qquad (3.60)$$
$$where\, q_k = a\tan\left(\frac{\mathrm{Im}\{C_k\}}{\mathrm{Re}\{C_k\}}\right)$$

- If $s_k < 0$ then the system is *unconditionally stable* for bounded inputs since the pole-zero map is on the left half plane as before.
- If $s_k > 0$ then the system is *generally unstable* since the pole-zero map is sometimes on the right half plane and other times on the LHP.

Case 2. Simple Poles on jw-axis:

In this case, pole-zero map passes through jw-axis; at that instant $s_k = 0$ and the pair of roots are purely imaginary and hence, the system is oscillatory with an impulse response:

$$h_k(t) = 2|C_k|.Cos(w_k t + q_k) \qquad (3.61)$$

Conclusion:

- For bounded input signals, the output signal will be bounded oscillatory or simply bounded. But if the input is also sinusoidal with the same w_k, the output will be of the form:

$$Y(s) = \frac{B.s}{(s^2 + w_k^2)^2} \quad \Leftrightarrow \quad y(t) = \frac{B}{2w_k}.t.Sin(w_k t) \qquad (3.62)$$

which will *grow unboundedly* as t is increases.

Case 3. Multiple-Order (Repeated) Poles in LHP of s-Plane:

Let us assume the order of the terms is *m*, then we will have an impulse response:

$$h_k(t) = |C_k|.t^{m-1}.e^{s_k t}.Cos(w_k t + q_k) \quad for \quad s_k < 0 \qquad (3.63)$$

Conclusion:

Since $e^{s_k t}$ will decay faster than the term t^{m-1} increases and the response in (3.63) will be *BIBO stable* if the input is bounded.

Case 4. Multiple-Order (Repeated) Poles on jw-axis:

Again let us assume the order of the terms is *m*, then we will have an impulse response:

$$h_k(t) = |C_k| . t^{m-1} . Cos(w_k t + q_k) \quad for \quad s_k < 0 \tag{3.64}$$

Conclusion:

Here t^{m-1} increases as t grows and the system will be *unstable*.

Case 5. Multiple-Order (Repeated) Poles in RHP of s-Plane:

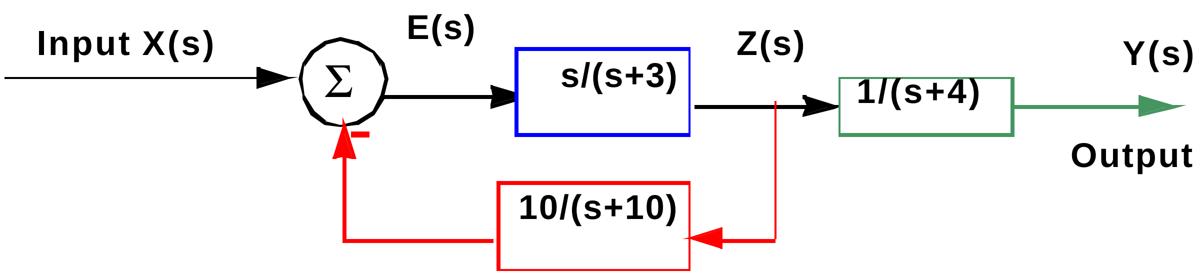
The response for these poles will be:

$$h_k(t) = |C_k| . t^{m-1} . e^{s_k t} . Cos(w_k t + q_k) \quad but \quad s_k > 0 \tag{3.65}$$

Conclusion:

Again, both $e^{s_k t}$ and t^{m-1} will grow as t increases. Expectedly, the system will be *unstable*.

Example 3.13: Determine if the following feedback system is stable.



The following set of equations governs this system.

$$E(S) = X(s) - Z(s).\frac{10}{s+10}$$

$$Z(s) = E(s).\frac{s}{s+3}$$

$$Y(s) = Z(s).\frac{1}{s+4}$$

The overall transfer function of this control system can be obtained as follows:

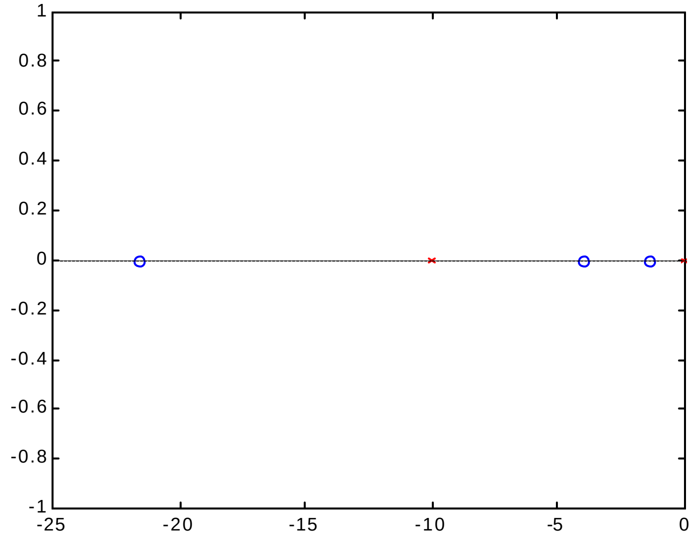
$$H_1(s) \equiv \frac{Z(s)}{X(s)} = \frac{s/(s+3)}{1 + \frac{s}{s+3}.\frac{10}{s+10}} = \frac{s^2 + 10s}{s^2 + 23s + 30}$$

and

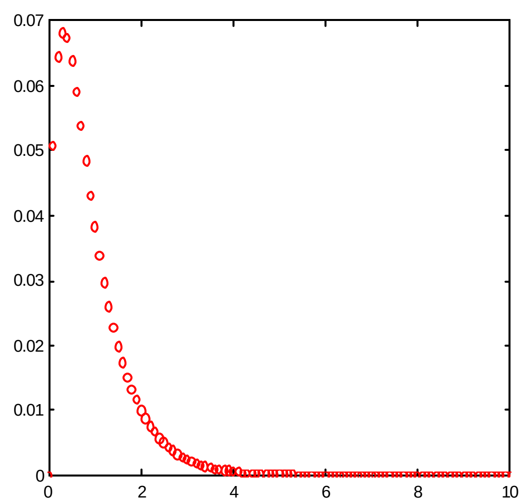
$$H(s) \equiv \frac{Y(s)}{X(s)} = \frac{Y(s)}{Z(s)}.\frac{Z(s)}{X(s)} = \frac{1}{s+4}.\frac{s^2 + 10s}{s^2 + 23s + 30} = \frac{s^2 + 10s}{s^3 + 27s^2 + 122s + 120}$$

As we have done in a few examples by now, the best way to finish this problem is to resort to Matlab tools.

```
%ILLUSTRATION OF EXAMPLE 3.13
b=[1,10,0];
a=[1,27,122,120];
%pole-zero map
pzmap(b,a);
pause;
```




```
%Step-Input Response
time=0:0.1:10;
response=step(b,a,time);
plot(time,response,'rO');
axis;
```



As we can see observe from the above pole-zero map, both zeros and poles are all real with $z_1=0;$ $z_2=-10$, $p_1=-1.388;$ $p_2=-4;$ *and* $p_3=-21,6$. Therefore, the system is stable for bounded inputs, which is clearly demonstrated in the step-response plot.