Chap 5. Continuous-Time Fourier Transform and Applications

5.1 Illustrative Definition of Fourier Transform

In this chapter, we will develop the basis for Fourier analysis of non-periodic signals, which is the only group of signals meaningful in engineering and real-life applications. Traditionally, Fourier analysis is presented by giving the definitions as we did for Laplace transforms in Chapter 3. Here, we will approach it as a limiting behavior of Fourier series analysis for periodic signals. To do that, let us consider an aperiodic (non-periodic) signal \( x(t) \) and form its periodic extension \( x_T(t) \) by repeating it every \( T \) seconds, where \( T \) is the period.

It is clear from above that the expansion signal can be written as:

\[
x_T(t) = \begin{cases} 
  x(t) & \text{if } 0 \leq t < T \\
  x(t) = x(t + nT) & \text{for every integer } n
\end{cases}
\]  

(5.1)

The Fourier series representation for this new periodic signal is simply:

**Synthesis Equation:** \( x_T(t) = \sum_{k=-\infty}^{\infty} F_k e^{jkw_0 t} \)  

(5.2a)

**Analysis Equation:** \( F_k = \frac{1}{T} \int_{<T>} x_T(t) e^{-jkw_0 t} \, dt \)  

(5.2b)

In the limit as \( T \to \infty \), we observe that \( w_0 = \frac{2\pi}{T} \to dw \), which is an infinitesimally small quantity in frequency-domain and it implies that \( kw_0 \to w \), a continuous variable and \( \frac{1}{T} \to \frac{dw}{2\pi} = df \). Finally, with this limiting behavior, the summation in (5.2a) becomes an integral as follows:

\[
F_k = \frac{dw}{2\pi} \int_{-\infty}^{\infty} x_T(t) e^{-jwt} \, dt \quad \Rightarrow \quad F_k = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_T(t) e^{-jwt} \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{-jwt} \, dt
\]

(5.3)

and

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} x(t) e^{-jwt} \, dt) e^{jwt} \, dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} \, dw
\]

(5.4)
With this result, we have implicitly defined the Fourier transform relationships for \( x(t) \).

**Analysis Equation:** 
\[
X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = F\{x(t)\} \tag{5.5a}
\]

**Synthesis Equation:** 
\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w) e^{jwt} dw = F^{-1}\{X(w)\} \tag{5.5b}
\]

These two equations are called Fourier transform pair and normally shown by: \( x(t) \Leftrightarrow X(w) \).

In general, \( X(w) \) is a complex function of the real-valued frequency variable \( w \) and it is written in terms of magnitude and phase terms:

\[
X(w) = |X(w)| e^{j\phi(w)} \tag{5.6}
\]

Therefore, we plot one curve for the magnitude and another one for the phase of a given expression.

**Dirichlet Conditions:** Fourier transform exists if:

1. \( x(t) \) is absolutely integrable: \( \int_{-\infty}^{\infty} |x(t)| dt < \infty \)
2. \( x(t) \) is a well-behaving function. That is, only a finite number of jumps of finite size, minima and maxima occur within any finite interval: \( t_1 < t < t_2 \).

**Example 5.1** Find the Fourier transform of a rectangular pulse (gate function, rectangular time-window).
\[ X(w) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} \, dt = \frac{e^{-j\omega \tau/2}}{-j\omega} \left[ e^{+j\omega \tau/2} - e^{-j\omega \tau/2} \right] = \frac{1}{-j\omega} \left[ -2j \sin \left( \frac{w\tau}{2} \right) = \frac{2}{w} \sin \left( \frac{w\tau}{2} \right) = \tau \sin \left( \frac{w\tau}{2\pi} \right) = \tau \text{Sa} \left( \frac{w\tau}{2} \right) \right] \]

- It is easy to observe that the resulting spectrum is real and symmetrical for all values of frequency \( w \).
- This implies that the phase response is zero: \( \phi(w) = 0 \) for all \( w \).
- The magnitude is decreasing as a function of \( 1/w \).
- Approximately, 90\% of energy content of this spectrum is under the main lobe of the plot: \(-2\pi/\tau < w < 2\pi/\tau\). In other words, the energy under all the tail lobes is about 10\%.

**Example 5.2** Find the Fourier transform of a delta function. We will solve this problem using the Sifting Theorem definition of a delta function:

\[ F\{V_0 \delta(t)\} = V_0 \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t} \, dt = V_0 \Rightarrow F\{\delta(t)\} = 1 \Rightarrow \delta(t) \leftrightarrow 1 \quad (5.9) \]

Similarly, we can use the inversion formula to have:

\[ \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \, dw \quad (5.10) \]

From this final result we can deduce that the Fourier transform of a constant is a delta function in frequency-domain:

\[ F\{V_0\} = 2\pi V_0 \delta(w) \quad (5.11) \]

**Example 5.3** Find the Fourier transform of a complex exponential harmonic function:

\[ F\{e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} e^{-j(\omega-w_0)t} \, dt = 2\pi \delta(w-w_0) \Rightarrow e^{j\omega_0 t} \leftrightarrow 2\pi \delta(w-w_0) \quad (5.12) \]

On the other hand, the Fourier transform of a one-sided decaying exponential function is:

\[ x(t) = e^{-at}u(t) \quad \text{and} \quad a > 0 \]

\[ X(w) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t} \, dt = \int_{0}^{\infty} e^{-(a+j\omega)t} \, dt = \frac{1}{a+j\omega} \quad (5.13) \]

**Example 5.4** Find the Fourier transform of a periodic signal \( x(t) \) with a period \( T_0 = \frac{2\pi}{w_0} \). As we know from the previous chapter that the periodic functions have Fourier series representation:

\[ x(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} \]

Let us take Fourier transform of this equation term-by-term:
\[ X(w) = F\{x(t)\} = \sum_{k=-\infty}^{\infty} F_k \cdot e^{jk\omega_0} = \sum_{k=-\infty}^{\infty} 2\pi.F_k \cdot \delta(w - kw_0) = \sum_{k=-\infty}^{\infty} 2\pi.F_k \cdot \delta(w - k\frac{2\pi}{T_0}) \]

(5.14)

Therefore, the Fourier transform is a sequence of impulse functions regardless of the actual shape of the signal \(x(t)\). In other words, the Fourier transform of all periodic functions is a family of impulses. What make them different for various \(x(t)\) shapes are the values of the coefficients \(\{F_k\}\).

**Example 5.5** Using the results from the previous example, we are asked to find the Fourier transform of an impulse train.

\[ x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) \]

\[ X(w) = \sum_{k=-\infty}^{\infty} 2\pi.F_k \cdot \delta(w - kw_0) = \sum_{k=-\infty}^{\infty} 2\pi.F_k \cdot \delta(w - k\frac{2\pi}{T_0}) \]

\[ F_k = \frac{1}{T_0} \cdot \int x(t).e^{-jk\omega_0} = \frac{1}{T_0} \cdot \int \delta(t - kT_0).e^{-jk\omega_0} = \frac{1}{T_0} \]

Let us substitute these coefficient values in to the result above to obtain:

\[ X(w) = \sum_{k=-\infty}^{\infty} 2\pi.T_0 \cdot \frac{1}{T_0} \cdot \delta(w - k\frac{2\pi}{T_0}) = \frac{2\pi}{T_0} \cdot \sum_{k=-\infty}^{\infty} \delta(w - k\frac{2\pi}{T_0}) \]

(5.15)

Thus, we conclude that the Fourier transform of an impulse train is another impulse train in the frequency-domain with different strengths in the coefficient set.

**5.2 Properties of Fourier Transforms**

1. **Linearity**: The Fourier transform is a linear transform.
   \[ a.f(t) + b.g(t) \leftrightarrow a.F(w) + b.G(w) \]  
   (5.17)

2. **Symmetry**: If \(x(t)\) is a real signal then \(X(-w) = X^*(w)\) or in polar form:
   \[ X(w) = |X(w)|e^{j\phi(w)} \quad \text{and} \quad X^*(w) = X(-w) = |X(w)|e^{-j\phi(w)} \]  
   (5.18)

Therefore, we have an even-symmetry of the amplitude spectrum and an odd-symmetry for the phase spectrum.
Example 5.6 Using the properties of real signals we can turn Fourier transform into a sine or a cosine transform. In particular, discrete version of the cosine transform has found a neat application area in image compression based on JPEG and MPEG techniques. These compression techniques and their derivatives attempt to exploit the symmetry properties of imagery in the frequency-domain.
\[ X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt = \int_{-\infty}^{\infty} x(t)[Cos(wt) - jSin(wt)] dt \]
\[ = \int_{-\infty}^{\infty} x(t)Cos(wt)dt - j \int_{-\infty}^{\infty} x(t)Sin(wt)dt \]
\[ = \int_{-\infty}^{\infty} x(t)Cos(wt)dt + 0 \quad (Sin \ is \ odd \ fn.) \]
\[ = 2\int_{0}^{\infty} x(t).Cos(wt)dt \]

As we see from this last result that the Fourier transform turns into a Fourier Cosine transform for real signals.

3. **Time-Shifting (Delays):** Given: \( x(t) \leftrightarrow X(w) \) then
\[ x(t - T) \leftrightarrow X(w)e^{-j\omega T} = |X(w)|e^{j(\phi - \omega T)} \] \hspace{1cm} (5.20)

**Notes:**
- Amplitude spectrum: \( |X(w)| \) is unaffected by a delay (time-shift) of \( T \) units.
- However, the phase spectrum is linearly shifted by \( -\omega T \) radians from the phase value \( \phi \) of the original spectrum \( X(w) \). This linear phase-shift property makes the time-delay issue very useful in many applications.

4. **Time-Scaling (Resolution Change):** Given the Fourier pair: \( x(t) \leftrightarrow X(w) \) then a time-scale change will correspond to:
\[ x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{w}{a}\right) \] \hspace{1cm} (5.21)

**Notes:**
- For values of \( a \) larger than "1" this works as a compressor. That is, it counts slower than one time-unit at a time.
- On the other hand, for \( a \) smaller than "1" it works as an expander to increase the resolution in time-domain. The resulting behavior in frequency-domain is just the opposite.

**Example 5.7** Using the above property find the Fourier transform of \( x(t) = a.\text{rect}\left(\frac{at}{T}\right) \)

Recalling \( F\{\text{rect}\left(\frac{t}{T}\right)\} = T.\text{Sinc}\left(\frac{wT}{2\pi}\right) \) and the resolution change we get:
\[ F\{a.\text{rect}\left(\frac{wT}{2a\pi}\right)\} = T.\sin c\left(\frac{wT}{2a\pi}\right) \] \hspace{1cm} (5.22)
5. **Derivatives and Integrals - Scaling**: Given the Fourier pair: \( x(t) \Leftrightarrow X(w) \) then we have:

a. \( \frac{d}{dt} x(t) \Leftrightarrow jwX(w) \)  

(5.23a)

b. \( \frac{d^n}{dt^n} x(t) \Leftrightarrow (jw)^nX(w) \)  

(5.23b)

c. \( \int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \pi X(0)\delta(w) + \frac{1}{jw}X(w) \)  

(5.23c)

d. If there is no D.C. term, i.e., \( X(0) = 0 \) then \( \int_{-\infty}^{t} x(\tau)d\tau \Leftrightarrow \frac{1}{jw}X(w) \)  

(5.23d)

**Example 5.8** Using the above property find the Fourier transform of a unit-step function: \( u(t) \).

![Unit-step function graph]

As we see from the above figure, the unit-step function can be written as:

\[
u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)
\]

(5.24)

But we also know that:

\[
\frac{d}{dt}\left\{\frac{1}{2}\text{sgn}(t)\right\} = \delta(t) \quad \text{equivalently:} \quad (jw)F\left\{\frac{1}{2}\text{sgn}(t)\right\} = 1
\]

(5.25)

which results in:

\[
F\left\{\frac{1}{2}\text{sgn}(t)\right\} = \frac{1}{jw} \quad \text{and} \quad F\left\{\frac{1}{2}\right\} = \frac{1}{2}2\pi\delta(w)
\]

(5.26)

\[
F\{u(t)\} = \pi\delta(w) + \frac{1}{jw}
\]

(5.27)

6. **Energy in Aperiodic Signals (Parseval's Theorem)**: Recalling energy in time-domain:

\[
E = \int_{-\infty}^{\infty} |x(t)|^2dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)e^{-jw}dw\right]dt
\]

\[
= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)\left[\int_{-\infty}^{\infty} x(t)e^{-jw}dt\right]dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(w)X(w)dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(w)|^2dw
\]

which implies that the energy is conserved during Fourier transform and it can be summarized by:
\[ E = \int_{-\infty}^{\infty} [x(t)]^2 dt = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w).X(w)dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)|^2 dw \quad (5.28) \]

Similarly, the energy content of a signal within a finite band of frequency: \( w_1 \leq w \leq w_2 \) will be:

\[ \Delta E_x = \frac{1}{2\pi} \int_{w_1}^{w_2} |X(w)|^2 dw \quad (5.29) \]

**Example 5.9** Compute the portion of the energy for an exponentially decaying signal in the frequency interval: \(-4 \leq w \leq 4\). We know the Fourier transform for this signal both from a previous example and the Fourier Tables 5.1 that:

\[ x(t) = e^{-t}u(t) \Rightarrow X(w) = \frac{1}{1+jw} \Rightarrow |X(w)|^2 = \frac{1}{1+w^2} \]

When we substitute this into (5.28) and (5.29) we get:

\[ E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+w^2}dw = \frac{1}{2} \Rightarrow \text{From integral tables.} \]

\[ \Delta E_x = \frac{1}{2\pi} \int_{-4}^{4} |X(w)|^2 dw = \frac{1}{\pi} \int_{0}^{\frac{\pi}{2}} \frac{1}{1+w^2}dw = \frac{1}{\pi} \arctan(w)|_0^4 \equiv 0.422 \]

Therefore, the portion of the energy in this finite band is 84.4%.

7. **Convolution Operation:**

\[ y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (5.30a) \]

\[ Y(w) = X(w).H(w) = |Y(w)|e^{i\phi(w)} \]

\[ |Y(w)| = |X(w)| |Y(w)| \quad \text{and} \quad \angle Y(w) = \angle X(w) + \angle H(w) \quad (5.30c) \]

**Notes:**
- Output magnitude spectrum is the **product** of magnitude spectrum of the input and the frequency response of the system.
- The phase spectrum is the **sum** of the phase spectra of the input and the system.

**Example 5.10** Compute the unit-step response to a system with: \( h(t) = e^{-at}u(t) \).

\[ Y(w) = X(w).H(w) = F\{x(t)\}.F\{h(t)\} \]
\[ Y(w) = \left[ \pi \delta(w) + \frac{1}{jw} \right], \quad \frac{1}{a + jw} = \frac{\pi \delta(w)}{a + jw} + \frac{1}{jw} \frac{1}{a + jw} \quad \text{directly from Fourier Tables} \]

Let us note that \( \delta(w) \neq 0 \) only when \( w = 0 \). Therefore, the first term in the last expression is simply: \( 1^{st} \text{ Term:} = \frac{\pi}{a} \delta(w) \). Using this and partial fraction expansion results in:

\[
Y(w) = \frac{\pi}{a} \delta(w) + \frac{A}{jw} + \frac{B}{a + jw}
\]

\[ Bjw + A(a + jw) = 1 \quad \Rightarrow \quad A = \frac{1}{a} \quad \text{and} \quad B = -\frac{1}{a} \]

Finally, the output in the frequency-domain will be:

\[
Y(w) = \frac{\pi}{a} \delta(w) + \frac{1/a}{jw} - \frac{1/a}{a + jw}
\]

and the inverse Fourier transform yields the answer:

\[ y(t) = \frac{1}{a} \{1 - e^{-at}\} u(t) \]

8. \textbf{Modulation Theorem:}

\[ y(t) = x(t).m(t) \quad \Leftrightarrow \quad Y(w) = \frac{1}{2\pi} \left[ X(w) * M(w) \right] \quad (5.31) \]

\[ Y(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\psi).H(\psi - w)dw \quad (5.32) \]

In other words, to replace a time-domain multiplication, we must perform a frequency-domain convolution. As expected, we will attempt to perform time-domain multiplication as much as we can to tackle modulation tasks.

\textbf{Example 5.11} Given a signal with a base-band spectrum (frequency content of the signal in its natural habitat), modulate it with an impulse train of period: \( T_0 = \frac{2\pi}{w_0} \).
The output of the modulator (mixer) is simply:

\[ x_S(t) = x(t), p(t) = x(t), \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} x(nT_0) \delta(t - nT_0) \]  

(5.33)

Here, the impulse train acts like a sampling function, i.e., isolating the signal at a given point. Equivalently, in the frequency-domain:

\[ P(w) = F\{ p(t) \} = F\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_0) \} = \frac{2\pi}{T_0} \sum_{w=-\infty}^{\infty} \delta(w - \frac{2\pi}{T_0} k) \]  

(5.34)

\[ X_S(w) = \frac{1}{2\pi} X(w) * P(w) = \frac{1}{2\pi} \cdot \frac{2\pi}{T_0} \sum_{w=-\infty}^{\infty} [X(w) * \delta(w - \frac{2\pi}{T_0} k)] = \frac{1}{T_0} \sum_{w=-\infty}^{\infty} X(w - \frac{2\pi}{T_0} k) \]  

(5.35)

It is clear from this result that the output spectrum is a periodic family of the input spectra shifted to the neighborhood of \( \pm kw_0 = \pm k \frac{2\pi}{T_0} \).
Table 5.2 Properties of Continuous-Time Fourier Pairs

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