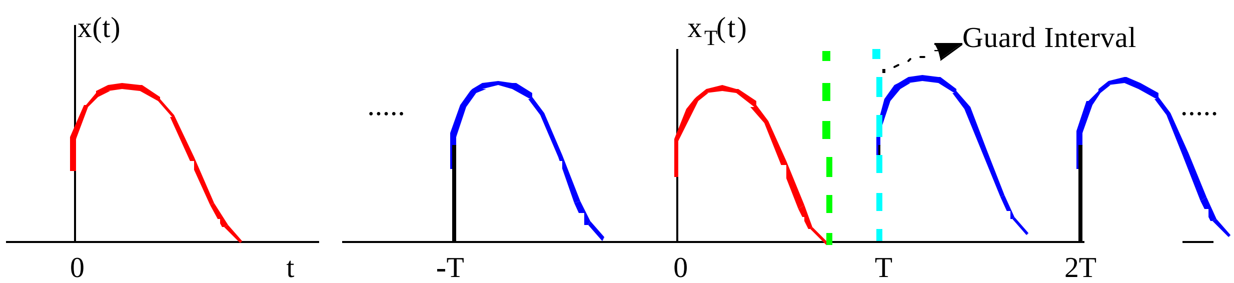
Chap 5. Continuous-Time Fourier Transform and Applications

5.1 Illustrative Definition of Fourier Transform

In this chapter, we will develop the basis for Fourier analysis of non-periodic signals, which is the only group of signals meaningful in engineering and real-life applications. Traditionally, Fourier analysis is presented by giving the definitions as we did for Laplace transforms in Chapter 3. Here, we will approach it as a limiting behavior of Fourier series analysis for periodic signals. To do that, let us consider an aperiodic (non-periodic) signal $x(t)$ and form its periodic extension $x_T(t)$ by repeating it every T seconds, where T is the period.



It is clear from above that the expansion signal can be written as:

$$x_T(t) = \begin{cases} x(t) & if\ \ 0 \le t < T \\ x(t) = x(t+nT) & for\ every\ \text{int}\ eger\ n \end{cases} \qquad (5.1)$$

The Fourier series representation for this new periodic signal is simply:

$$Synthesis\ Equation: \quad x_T(t) = \sum_{k=-\infty}^{\infty} F_k . e^{jkw_0 t} \qquad (5.2a)$$

$$Analysis\ Equation: \quad F_k = \frac{1}{T} . \int_{<T>} x_T(t).e^{-jkw_0 t} dt \qquad (5.2b)$$

In the limit as $T \Rightarrow \infty$, we observe that $w_0 = \frac{2p}{T} \Rightarrow dw$, which is an infinitesimally small quantity in frequency-domain and it implies that $kw_0 \Rightarrow w$, a continuous variable and $\frac{1}{T} \Rightarrow \frac{dw}{2p} = df$. Finally, with this limiting behavior, the summation in (5.2a) becomes an integral as follows:

$$F_k = \frac{dw}{2p} . \int_{-\infty}^{\infty} x_T(t).e^{-jwt} dt \quad \Rightarrow \quad \frac{F_k}{dw} = \frac{1}{2p} \int_{-\infty}^{\infty} x_T(t).e^{-jwt} dt = \frac{1}{2p} \int_{-\infty}^{\infty} x(t).e^{-jwt} dt \qquad (5.3)$$

and

$$x(t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(t).e^{-jwt} dt \right).e^{jwt} \frac{dw}{2p} = \frac{1}{2p} \int_{-\infty}^{\infty} X(w).e^{jwt} dw \qquad (5.4)$$

With this result, we have implicitly defined the Fourier transform relationships for x(t).

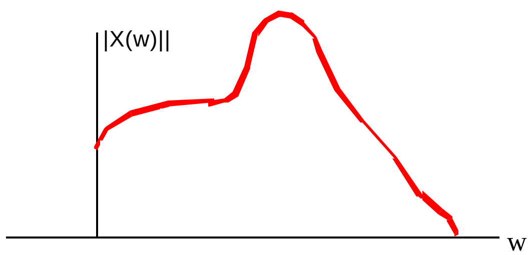
$$Analysis\ Equation: \quad X(w) = \int_{-\infty}^{\infty} x(t).e^{-jwt}dt = F\{x(t)\} \qquad (5.5a)$$

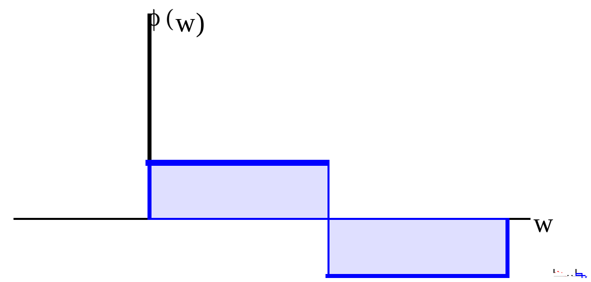
$$Synthesis\ Equation: \quad x(t) = \frac{1}{2p}\int_{-\infty}^{\infty} X(w).e^{jwt}dw = F^{-1}\{X(w)\} \qquad (5.5b)$$

These two equations are called Fourier transform pair and normally shown by: $x(t) \Leftrightarrow X(w)$.
In general, X(w) is a complex function of the real-valued frequency variable w and it is written in terms of magnitude and phase terms:

$$X(w) = |X(w)|.e^{jf(w)} \qquad (5.6)$$

Therefore, we plot one curve for the magnitude and another one for the phase of a given expression.

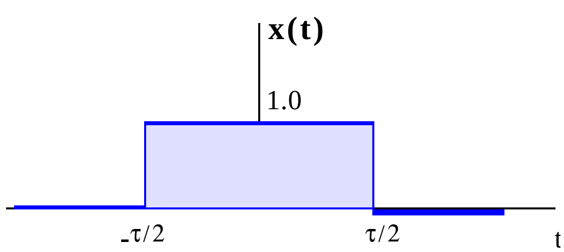


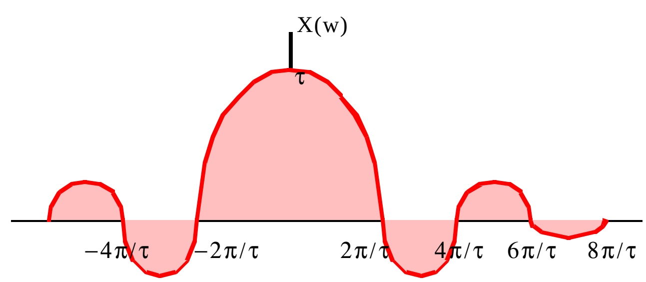


Dirichlet Conditions: Fourier transform exists if:

1. *x(t)* is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
2. *x(t)* is a *well-behaving* function. That is, only a finite number of jumps of finite size, minima and maxima occur within any finite interval: $t_1 < t < t_2$.

Example 5.1 Find the Fourier transform of a rectangular pulse (gate function, rectangular time-window).





$$X(w) = \int_{-\infty}^{\infty} x(t).e^{-jwt}dt = \int_{-t/2}^{t/2} 1.e^{-jwt}dt = \left.\frac{e^{-jwt}}{-jw}\right|_{t=-t/2}^{t/2} = \frac{1}{-jw}.[e^{-jwt/2} - e^{+jwt/2}]$$

$$= \frac{1}{-jw}.[-2j.Sin(\frac{wt}{2})] = \frac{2}{w}.Sin(\frac{wt}{2}) = t.Sinc(\frac{wt}{2p}) = t.Sa(\frac{wt}{2}) \qquad (5.7)$$

- It is easy to observe that the resulting spectrum is real and symmetrical for all values of frequency *w*.
- This implies that the phase response is zero: $f(w) = 0 \quad for\ all\ w.$
- The magnitude is decreasing as a function of $1/w$.
- Approximately, 90% of energy content of this spectrum is under the *main lobe* of the plot: $-2p/t < w < 2p/t$. In other words, the energy under all the *tail lobes* is about 10%.

Example 5.2 Find the Fourier transform of a delta function. We will solve this problem using the Sifting Theorem definition of a delta function:

$$F\{V_0 d(t)\} = V_0 \int_{-\infty}^{\infty} d(t).e^{-jwt}dt = V_0 \quad \Rightarrow \quad F\{d(t)\} = 1 \quad \Rightarrow \quad d(t) \Leftrightarrow 1 \qquad (5.9)$$

Similarly, we can use the inversion formula to have:

$$d(t) = \frac{1}{2p} \int_{-\infty}^{\infty} 1.e^{jwt}dw \qquad (510)$$

From this final result we can deduce that the Fourier transform of a constant is a delta function in frequency-domain:

$$F\{V_0\} = 2p.V_0.d(w) \qquad (5.11)$$

Example 5.3 Find the Fourier transform of a complex exponential harmonic function:

$$F\{e^{jw_0t}\} = \int_{-\infty}^{\infty} e^{jw_0t}.e^{-jwt}dt = \int_{-\infty}^{\infty} e^{-j(w-w_0)t}dt = 2pd(w-w_0) \qquad e^{jw_0t} \Leftrightarrow 2pd(w-w_0) \qquad (5.12)$$

On the other hand, the Fourier transform of a one-sided decaying exponential function is:

$$x(t) = e^{-at}u(t) \quad and \quad a > 0$$

$$X(w) = \int_{-\infty}^{\infty} e^{-at}.u(t).e^{-jwt}dt = \int_{0}^{\infty} 1.e^{-(a+jw)t}dt = \frac{1}{a+jw} \qquad (5.13)$$

Example 5.4 Find the Fourier transform of a periodic signal x(t) with a period $T_0 = \frac{2p}{w_0}$. As we know from the previous chapter that the periodic functions have Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{\infty} F_k.e^{jkw_0t}$$

Let us take Fourier transform of this equation term-by-term:

$$X(w) = F\{x(t)\} = \sum_{k=-\infty}^{\infty} F_k . F\{e^{jkw_0 t}\} = \sum_{k=-\infty}^{\infty} 2p.F_k\, d(w - kw_0) = \sum_{k=-\infty}^{\infty} 2p.F_k\, d(w - k\frac{2p}{T_0}) \quad (5.14)$$

Therefore, the Fourier transform is a sequence of impulse functions regardless of the actual shape of the signal *x(t)*. In other words, the Fourier transform of all periodic functions is a family of impulses. What make them different for various *x(t)* shapes are the values of the coefficients $\{F_k\}$.

Example 5.5 Using the results from the previous example, we are asked to find the Fourier transform of an impulse train.

$$x(t) = \sum_{k=-\infty}^{\infty} d(t - kT_0)$$

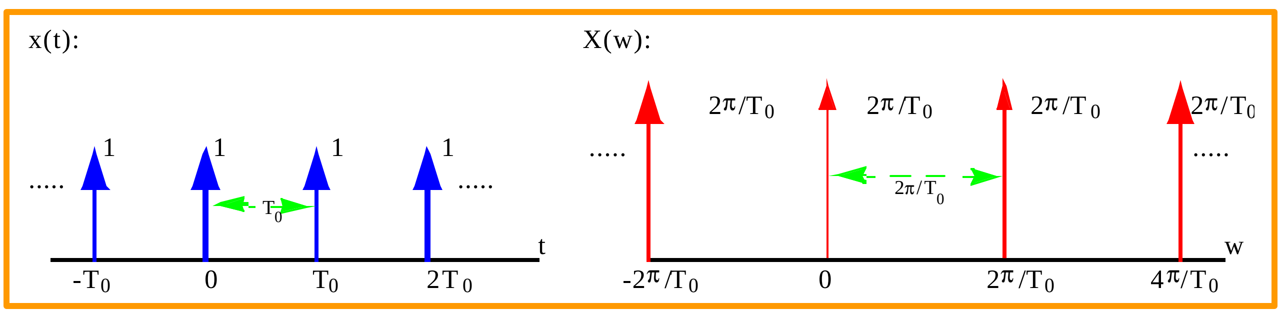
$$X(w) = \sum_{k=-\infty}^{\infty} 2p.F_k\, d(w - kw_0) = \sum_{k=-\infty}^{\infty} 2p.F_k\, d(w - k\frac{2p}{T_0})$$

$$F_k = \frac{1}{T_0}.\int_{<T_0>} x(t).e^{-jkw_0 t} = \frac{1}{T_0}.\int_{<T_0>} d(t - kT_0).e^{-jkw_0 t} = \frac{1}{T_0} \quad (5.15)$$

Let us substitute these coefficient values in to the result above to obtain:

$$X(w) = \sum_{k=-\infty}^{\infty} 2p.\frac{1}{T_0} d(w - k\frac{2p}{T_0}) = \frac{2p}{T_0}.\sum_{k=-\infty}^{\infty} .d(w - k\frac{2p}{T_0}) \quad (5.16)$$

Thus, we conclude that the Fourier transform of an impulse train is another impulse train in the frequency-domain with different strengths in the coefficient set.



5.2 Properties of Fourier Transforms

1. **Linearity:** The Fourier transform is a linear transform.

$$a.f(t) + bg(t) \Leftrightarrow a.F(w) + bG(w) \quad (5.17)$$

2. **Symmetry:** If *x(t)* is a real signal then $X(-w) = X^*(w)$ or in polar form:

$$X(w) = |X(w)|.e^{jf(w)} \quad and \quad X^*(w) = X(-w) = |X(w)|.e^{-jf(w)} \quad (5.18)$$

Therefore, we have an even-symmetry of the amplitude spectrum and an odd-symmetry for the phase spectrum.

Table 5.1 Fourier Transform Pairs for Selected Signals

1. 1	$2\pi\delta(\omega)$
2. $u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
3. $\delta(t)$	1
4. $\delta(t - t_0)$	$\exp[-j\omega t_0]$
5. $\text{rect}(t/\tau)$	$\tau \,\text{sinc}\,\frac{\omega\tau}{2\pi} = \frac{2\sin\omega\tau/2}{\omega}$
6. $\frac{\omega_B}{\pi}\,\text{sinc}\,\frac{\omega_B t}{\pi} = \frac{\sin\omega_B t}{\pi t}$	$\text{rect}(\omega/2\omega_B)$
7. $\text{sgn}\, t$	$\frac{2}{j\omega}$
8. $\exp[j\omega_0 t]$	$2\pi\delta(\omega - \omega_0)$
9. $\sum_{n=-\infty}^{\infty} a_n \exp[jn\omega_0 t]$	$2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$
10. $\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
11. $\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
12. $(\cos\omega_0 t)u(t)$	$\frac{\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$
13. $(\sin\omega_0 t)u(t)$	$\frac{\pi}{2j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \frac{\omega_0}{\omega_0^2 - \omega^2}$
14. $\cos\omega_0 t\ \text{rect}(t/\tau)$	$\tau\,\text{sinc}\,\frac{(\omega - \omega_0)\tau}{2\pi}$
15. $\exp[-at]u(t)$, $\text{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$
16. $t\exp[-at]u(t)$, $\text{Re}\{a\} > 0$	$\left(\frac{1}{a + j\omega}\right)^2$
17. $\frac{t^{n-1}}{(n-1)!}\exp[-at]u(t)$, $\text{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$
18. $\exp[-a\|t\|]$, $a > 0$	$\frac{2a}{a^2 + \omega^2}$
19. $\|t\|\exp[-a\|t\|]$, $\text{Re}\{a\} > 0$	$\frac{4aj\omega}{a^2 + \omega^2}$
20. $\frac{1}{a^2 + t^2}$, $\text{Re}\{a\} > 0$	$\frac{\pi}{a}\exp[-a\|\omega\|]$
21. $\frac{t}{a^2 + t^2}$, $\text{Re}\{a\} > 0$	$\frac{-j\pi\omega\exp[-a\|\omega\|]}{2a}$
22. $\exp[-at^2]$, $a > 0$	$\sqrt{\frac{\pi}{a}}\exp\left[\frac{-\omega^2}{4a}\right]$
23. $\Delta(t/\tau)$	$\tau\,\text{sinc}^2\,\frac{\omega\tau}{2\pi}$
24. $\sum_{n=-\infty}^{\infty}\delta(t - nT)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2n\pi}{T}\right)$

Example 5.6 Using the properties of real signals we can turn Fourier transform into a sine or a cosine transform. In particular, discrete version of the cosine transform has found a neat application area in image compression based on JPEG and MPEG techniques. These compression techniques and their derivatives attempt to exploit the symmetry properties of imagery in the frequency-domain.

$$
\begin{aligned}
X(w) &= \int_{-\infty}^{\infty} x(t).e^{-jwt}dt = \int_{-\infty}^{\infty} x(t).[Cos(wt) - jSin(wt)]dt \\
&= \int_{-\infty}^{\infty} x(t)Cos(wt)dt - j\int_{-\infty}^{\infty} x(t)Sin(wt)dt \\
&= \int_{-\infty}^{\infty} x(t)Cos(wt)dt + 0 \quad (Sin\ is\ odd\ fn.) \\
&= 2\int_{0}^{\infty} x(t).Cos(wt)dt
\end{aligned} \tag{5.19}
$$

As we see from this last result that the Fourier transform turns into a Fourier Cosine transform for real signals.

3. **Time-Shifting (Delays):** Given: $x(t) \Leftrightarrow X(w)$ then

$$x(t-T) \Leftrightarrow X(w).e^{-jwT} = |X(w)|.e^{j(f-wT)} \tag{5.20}$$

Notes:

- Amplitude spectrum: $|X(w)|$ is unaffected by a delay (time-shift) of T units.
- However, the phase spectrum is linearly shifted by $-wT$ radians from the phase value f of the original spectrum *X(w)*. This linear phase-shift property makes the time-delay issue very useful in many applications.

4. **Time-Scaling (Resolution Change):** Given the Fourier pair: $x(t) \Leftrightarrow X(w)$ then a time-scale change will correspond to:

$$x(at) \quad \Leftrightarrow \quad \frac{1}{|a|}.X(\frac{w}{a}) \tag{5.21}$$

Notes:

- For values of *a* larger than *"1"* this works as a compressor. That is, it counts slower than one time-unit at a time.
- On the other hand, for *a* smaller than *"1"* it works as an expander to increase the resolution in time-domain. The resulting behavior in frequency-domain is just the opposite.

Example 5.7 Using the above property find the Fourier transform of $x(t) = a.rect(\frac{at}{T})$

Recalling $F\{rect(\frac{t}{T})\} = T.Sinc(\frac{wT}{2p})$ and the resolution change we get:

$$F\{a.rect(\frac{wT}{2ap}) = T.\sin c(\frac{wT}{2ap}) \tag{5.22}$$

5. **Derivatives and Integrals:-Scaling:** Given the Fourier pair: $x(t) \Leftrightarrow X(w)$ then we have:

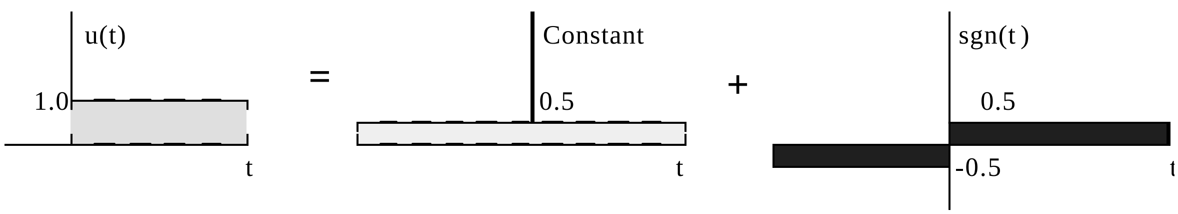
a. $$\frac{d}{dt}x(t) \quad \Leftrightarrow \quad jw.X(w) \tag{5.23a}$$

b. $$\frac{d^n}{dt^n}x(t) \quad \Leftrightarrow \quad (jw)^n.X(w) \tag{5.23b}$$

c. $$\int_{-\infty}^{t} x(t)dt \quad \Leftrightarrow \quad p.X(0).d(w)+\frac{1}{jw}.X(w) \tag{5.23c}$$

d. If there is no D.C. term, i.e., $X(0)=0$ then $\int_{-\infty}^{t} x(t)dt \quad \Leftrightarrow \quad \frac{1}{jw}.X(w)$ (5.23d)

Example 5.8 Using the above property find the Fourier transform of a unit-step function: *u*(*t*).



As we see from the above figure, the unit-step function can be written as:

$$u(t)=\frac{1}{2}+\frac{1}{2}.Sgn(t) \tag{5.24}$$

But we also know that:

$$\frac{d}{dt}\{\frac{1}{2}.Sgn(t)\}=d(t) \quad equivalently: \quad (jw).F\{\frac{1}{2}.Sgn(t)\}=1 \tag{5.25}$$

which results in:

$$F\{\frac{1}{2}.Sgn(t)\}=\frac{1}{jw} \quad and \quad F\{\frac{1}{2}\}=\frac{1}{2}.2p.d(w) \tag{5.26}$$

$$F\{u(t)\}=p.d(w)+\frac{1}{jw} \tag{5.27}$$

6. **Energy in Aperiodic Signals (Parseval's Theorem):** Recalling energy in time-domain:

$$E=\int_{-\infty}^{\infty}|x(t)|^2dt=\int_{-\infty}^{\infty}x(t).x^*(t)dt=\int_{-\infty}^{\infty}x(t).[\frac{1}{2p}\int_{-\infty}^{\infty}X^*(w).e^{-jw}dw]dt$$

$$=\frac{1}{2p}\int_{-\infty}^{\infty}X^*(w).[\int_{-\infty}^{\infty}x(t).e^{-jw}dt]dw=\frac{1}{2p}\int_{-\infty}^{\infty}X^*(w).X(w)dw=\frac{1}{2p}\int_{-\infty}^{\infty}|X(w)|^2dw$$

which implies that the energy is conserved during Fourier transform and it can be summarized by:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t).x^*(t)dt = \frac{1}{2p}\int_{-\infty}^{\infty} X^*(w).X(w)dw = \frac{1}{2p}\int_{-\infty}^{\infty} |X(w)|^2 dw \qquad (5.28)$$

Similarly, the energy content of a signal within a finite band of frequency: $w_1 \le w \le w_2$ will be:

$$\Delta E_x = \frac{1}{2p}\int_{w_1}^{w_2} |X(w)|^2 dw \qquad (5.29)$$

Example 5.9 Compute the portion of the energy for an exponentially decaying signal in the frequency interval: $-4 \le w \le 4$. We know the Fourier transform for this signal both from a previous example and the Fourier Tables 5.1 that:

$$x(t) = e^{-t}.u(t) \quad \Rightarrow \quad X(w) = \frac{1}{1+jw} \quad \Rightarrow \quad |X(w)|^2 = \frac{1}{1+w^2}$$

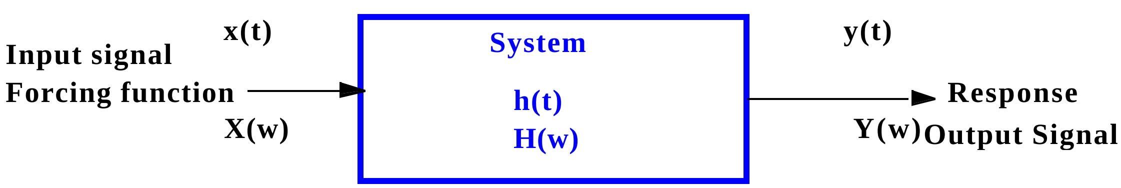
When we substitute this into (5.28) and (5.29) we get:

$$E = \frac{1}{2p}\int_{-\infty}^{\infty} \frac{1}{1+w^2} dw = \frac{1}{2} \quad \Rightarrow From\ \text{int}\ egration\ tables.$$

$$\Delta E_x = \frac{1}{2p}\int_{-4}^{4} |X(w)|^2 dw = \frac{1}{p}\int_{0}^{4} \frac{1}{1+w^2} dw = \frac{1}{p}.\arctan(w)\Big|_0^4 \cong 0.422$$

Therefore, the portion of the energy in this finite band is 84.4%.

7. Convolution Operation:



$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(t)h(t-t)dt \qquad (5.30a)$$

$$Y(w) = X(w).H(w) = |Y(w)|.e^{jf(w)} \qquad (5.30b)$$

$$|Y(w)| = |X(w)|.|Y(w)| \quad and \quad \angle Y(w) = \angle X(w) + \angle H(w) \qquad (5.30c)$$

Notes:

- Output magnitude spectrum is the ***product*** of magnitude spectrum of the input and the frequency response of the system.
- The phase spectrum is the ***sum*** of the phase spectra of the input and the system.

Example 5.10 Compute the unit-step response to a system with: $h(t) = e^{-at}.u(t)$.

$$Y(w) = X(w).H(w) = F\{x(t)\}.F\{h(t)\}$$

$$Y(w) = [p.d(w) + \frac{1}{jw}].\frac{1}{a+jw} = \frac{p.d(w)}{a+jw} + \frac{1}{jw}\frac{1}{a+jw} \quad \textit{directly from Fourier Tables}$$

Let us note that $d(w) \neq 0 \quad only\ when\ w = 0$. Therefore, the first term in the last expression is simply: $1^{st} Term = \frac{p}{a} d(w)$. Using this and partial fraction expansion results in:

$$Y(w) = \frac{p}{a} d(w) + \frac{A}{jw} + \frac{B}{a+jw}$$

$$Bjw + A(a + jw) = 1 \quad \Rightarrow \quad A = \frac{1}{a} \quad and \quad B = -\frac{1}{a}$$

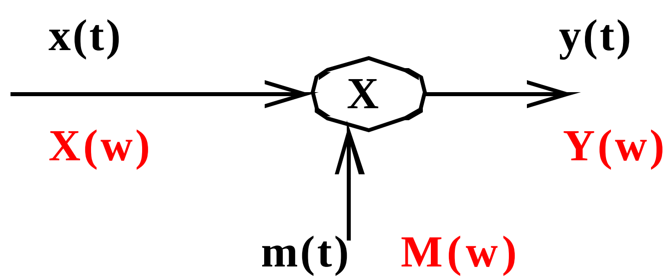
Finally, the output in the frequency-domain will be:

$$Y(w) = \frac{p}{a} d(w) + \frac{1/a}{jw} - \frac{1/a}{a+jw}$$

and the inverse Fourier transform yields the answer:

$$y(t) = \frac{1}{a}.\{1 - e^{-at}\}.u(t)$$

8. **Modulation Theorem:**

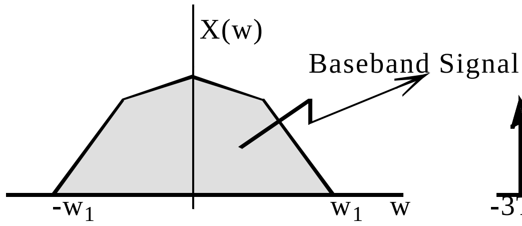


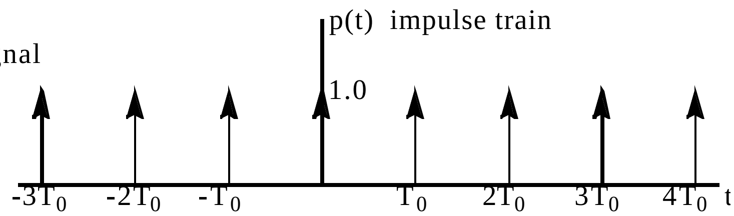
$$y(t) = x(t).m(t) \qquad \Leftrightarrow \qquad Y(w) = \frac{1}{2p}[X(w) * M(w)] \qquad (5.31)$$

$$Y(w) = \frac{1}{2p}\int_{-\infty}^{\infty} X(y).H(y - w)dw \qquad (5.32)$$

In other words, to replace a time-domain multiplication, we must perform a frequency-domain convolution. As expected, we will attempt to perform time-domain multiplication as much as we can to tackle modulation tasks.

Example 5.11 Given a signal with a base-band spectrum (frequency content of the signal in its natural habitat), modulate it with an impulse train of period: $T_0 = 2p / w_0$.





The output of the modulator (mixer) is simply:

$$x_S(t) = x(t).p(t) = x(t).\sum_{n=-\infty}^{\infty} d(t - nT_0) = \sum_{n=-\infty}^{\infty} x(nT_0).d(t - nT_0) \qquad (5.33)$$

Here, the impulse train acts like a sampling function, i.e., isolating the signal at a given point. Equivalently, in the frequency-domain:

$$P(w) = F\{p(t)\} = F\{\sum_{n=-\infty}^{\infty} d(t - nT_0)\} = \frac{2p}{T_0}.\sum_{w=-\infty}^{\infty} d(w - \frac{2p}{T_0}k) \qquad (5.34)$$

$$X_S(w) = \frac{1}{2p} X(w) * P(w) = \frac{1}{2p}.\frac{2p}{T_0}.\sum_{w=-\infty}^{\infty} [X(w) * d(w - \frac{2p}{T_0}k)] = \frac{1}{T_0}.\sum_{w=-\infty}^{\infty} X(w - \frac{2p}{T_0}k)$$

(5.35)

It is clear from this result that the output spectrum is a periodic family of the input spectra shifted to the neighborhood of $\pm kw_0 = \pm k\frac{2p}{T_0}$.

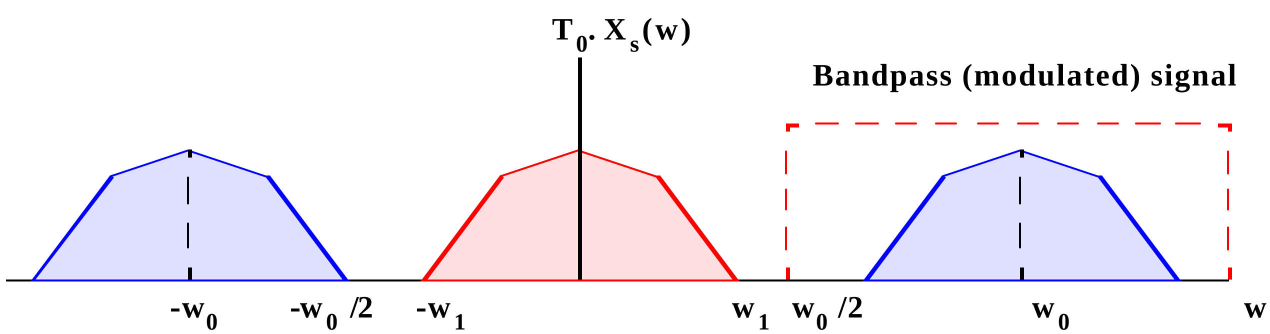


Table 5.2 Properties of Continuous-Time Fourier Pairs

1. Linearity	$\sum_{n=1}^{N} \alpha_n x_n(t)$	$\sum_{n=1}^{N} \alpha_n X_n(\omega)$
2. Complex conjugation	$x^*(t)$	$X^*(-\omega)$
3. Time shift	$x(t - t_0)$	$X(\omega) \exp[-j\omega t_0]$
4. Frequency shift	$x(t) \exp[j\omega_0 t]$	$X(\omega - \omega_0)$
5. Time scaling	$x(at)$	$1/\|a\| X(\omega/a)$
6. Differentiation	$d^n x(t)/dt^n$	$(j\omega)^n X(\omega)$
7. Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j(\omega)} + \pi X(0)\delta(\omega)$
8. Parseval's relation	$\int_{-\infty}^{\infty} \|x(t)\|^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} \|X(\omega)\|^2 d\omega$
9. Convolution	$x(t)*h(t)$	$X(\omega)H(\omega)$
10. Duality	$X(t)$	$2\pi\, x(-\omega)$
11. Multiplication by t	$(-jt)^n x(t)$	$\frac{d^n X(\omega)}{d\omega^n}$
12. Modulation	$x(t)m(t)$	$\frac{1}{2\pi} X(\omega) * M(\omega)$