Chapter on Fourier Series

Given a signal: $x(t) = x(t + nT_0); \quad for\ n = 0,1,2,3,\cdots$, where the fundamental period is $T_0 = \frac{2p}{w_0} = \frac{1}{f_0}$, the fundamental frequency and the fundamental radian frequency are: f_0 and w_0, respectively.

Continuous Signal Representation Conjecture:
Any periodic signal x(t) can be represented by an orthogonal set of basis functions, such as complex exponentials, pulse functions, and sinusoidal functions.

Example 4.1 Complex exponential basis functions.

$$f_k(t) = e^{jkw_0 t} = e^{j2kpf_0 t} = e^{jk\frac{2p}{T_0}t} \tag{4.1}$$

where k is the harmonic number. With these harmonics we can represent any *x(t)* by means of a *synthesis equation*:

$$x(t) = \sum_{k=-\infty}^{\infty} F_k . e^{jkw_0 t} \tag{4.2}$$

the coefficient set $\{F_k\}$ is called the synthesizer weights. In order to find these weights a procedure commonly known as the *analysis equation*:

$$F_k = \frac{1}{T_0}\int_0^{T_0} x(t).e^{-jkw_0 t}dt = \sum_{k=-\infty}^{\infty} x(t).e^{jkw_0 t} \tag{4.3}$$

Alternative Forms of Fourier Series:

- $$x(t) = \sum_{k=-\infty}^{\infty} F_k.e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} |F_k|.e^{j\angle F_k}.e^{jkw_0 t} = \sum_{k=-\infty}^{\infty} |F_k|.e^{j(kw_0 t + \angle F_k)}$$
  $$= \sum_{k=-\infty}^{\infty} F_k.e^{jk\frac{2p}{T_0}t} = \sum_{k=-\infty}^{\infty} F_k.e^{j2pkf_0 t} \tag{4.4}$$

- If the signal is real-valued, i.e., no imaginary portion, then we have the following property:

$$x(t) = \sum_{k=-\infty}^{\infty} F_k.e^{jkw_0 t} = F_0 + \sum_{k=1}^{\infty} F_k.e^{jkw_0 t} + \sum_{k=-\infty}^{-1} F_k.e^{jkw_0 t} \tag{4.5}$$

Let us make a change of variable in the last summation: *r=-k* and we now have:

$$x(t) = F_0 + \sum_{k=1}^{\infty} F_k.e^{jkw_0t} + \sum_{r=1}^{\infty} F_{-r}.e^{-jrw_0t}$$

Let us make one more change of variable: $k=r$ and the result is:

$$x(t) = F_0 + \{\sum_{k=1}^{\infty} F_k.e^{jkw_0t} + \sum_{k=1}^{\infty} F_{-k}.e^{-jkw_0t}\} = F_0 + 2.\sum_{k=1}^{\infty} \mathrm{Re}\{F_k.e^{jkw_0t}\}$$

$$= F_0 + 2.\sum_{k=1}^{\infty} \mathrm{Re}\{F_k\}.Cos(kw_0t) - 2.\sum_{k=1}^{\infty} \mathrm{Im}\{F_k\}.Sin(kw_0t) \quad (4.6)$$

With a set of coefficient assignment the last equation becomes:

$$x(t) = a_0 + 2.\sum_{k=1}^{\infty}[a_k.Cos(kw_0t) - b_k.Sin(kw_0t)] \quad (4.7)$$

It is not difficult to see that (2.6) and (2.7) are equivalent if:

$$a_0 = F_0 = D.C.Term = Time\ Average\ of\ \ x(t) = \frac{1}{T_0}.\int_{<T_0>} x(t)dt \quad (4.8a)$$

$$a_k = 2.\mathrm{Re}\{F_k\} = Cos\ terms\ with\ even\ symmetry = \frac{2}{T_0}.\int_{<T_0>} x(t).Cos(kw_0t)dt \quad (4.8b)$$

$$b_k = -2.\mathrm{Im}\{F_k\} = Sine\ term\ with\ odd\ symmetry = \frac{2}{T_0}.\int_{<T_0>} x(t).Sin(kw_0t)dt \quad (4.8c)$$

Here the notation of the limits of integration: $< T_0 >$ represents a one full period of T_0, regardless of the beginning point.

- $$x(t) = F_0 + \sum_{k=1}^{\infty} 2|F_k|Cos(kw_0t + f_k) = A_0 + \sum_{k=1}^{\infty} A_k.Cos(kw_0t + f_k)$$

  $$where\quad A_k = 2|F_k| \quad and \quad f_k = \angle F_k = \arg(F_k)$$

  (4.9)

Dirichlet Conditions For Existence of Fourier Series:

In order the Fourier analysis to exist if all of the following three conditions must hold:

1. *x(t)* has a finite number of maxima and minima (extrema) in one full period T_0.
2. *x(t)* has a finite number of discontinuities in one full period T_0.
3. *x(t)* is absolutely integrable (convergent) in one full period T_0.

$$\int_{<T_0>} |x(t)|dt < \infty \quad (4.10)$$

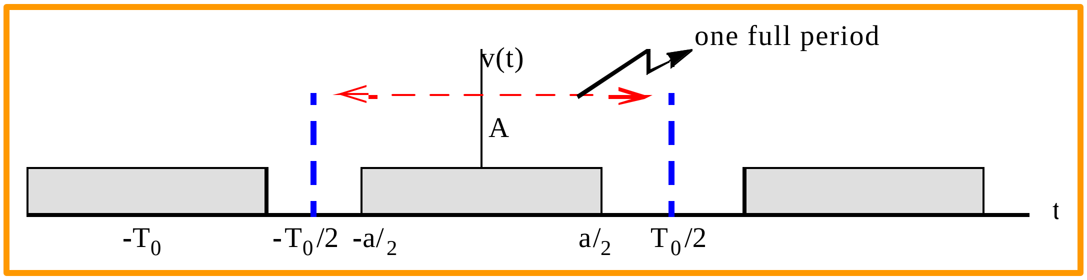
Example 4.2 Pulse trains, a periodic sequence of pulses, are frequently used in laboratories to test the design under study. Here we will compute the Fourier series for the following pulse train.

$$v(t) = v(t + kT_0); \quad T_0 = Period; \quad -\infty < k < \infty$$

But in each period, (why not the first one!) we have: $v(t) = \begin{cases} A & if \ -a/2 < t < a/2 \\ 0 & Outside \end{cases}$ (4.11)

Let us use the definition of Fourier coefficients above to compute:

$$F_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} A.e^{-jkw_0 t} dt = \frac{A}{T_0} \int_{-a/2}^{a/2} e^{-jkw_0 t} d = \frac{A}{T_0} . \frac{e^{-jkw_0 a/2} - e^{jkw_0 a/2}}{-jkw_0} \qquad (4.12)$$



To simplify the last result we use the trigonometric identity: $Sin(x) = \frac{1}{2j}(e^{jx} - e^{-jx})$ and

$$F_k = \frac{A}{T_0} . \frac{2}{kw_0} Sin(kw_0 \frac{a}{2}) = \frac{A}{2p/w_0} . \frac{2}{kw_0} Sin(kw_0 \frac{a}{2})$$

$$= \frac{A}{kp} Sin(kw_0 \frac{a}{2}) = \frac{A}{kp} Sin(kp \frac{a}{T_0}) \qquad (4.13)$$

Let us separate the D.C. coefficient from the rest:

$$F_0 = \frac{A}{T_0} . \int_{-a/2}^{a/2} 1.dt = \frac{Aa}{T_0} \qquad (4.14)$$

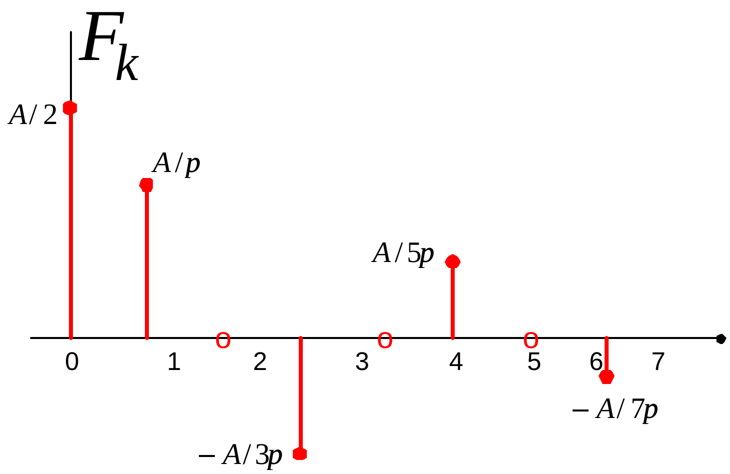
To illustrate the significance of these coefficients let us assume that $\frac{Aa}{T_0} = 0.5 = \%50 \ duty \ cycle = \% \ on - time / \ period$. We next compute the first few coefficients:

$$F_0 = A/2 \qquad F_1 = \frac{A}{p} Sin(p/2) = A/p \qquad F_2 = \frac{A}{2p} Sin(p) = 0 = F_4 = F_6 = \cdots = F_{2k}$$

$$F_3 = \frac{A}{3p} Sin(3p/2) = -A/3p \qquad F_5 = \frac{A}{5p} Sin(5p/2) = A/5p$$

$$F_7 = \frac{A}{7p} Sin(7p/2) = -A/7p$$

It is clear from above that all even coefficients are zero and the odd ones is inversely proportional to k.



Finally, we substitute the coefficients in (4.14) and (4.13) to the synthesis equation:

$$v(t) = \frac{Aa}{T_0} + \sum_{k=1}^{\infty} \frac{A}{kp} Sin(\frac{kpa}{T_0}).e^{jkw_0 t} \qquad (4.15)$$

Let us now illustrate this result for three different N to see what happens as N goes to infinity.

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%Example 4.2 Pulse Train Fourier Series Representation for Different N.
t=-3*pi:.05:3*pi;x = zeros(size(t))
for loop=1:50:101;
   loopend=loop*2; x = zeros(size(t));
   for k=1:1:loopend;
      x = x + (2/(k*pi))*sin(0.5*k)*cos(k*t);
   end;
   v=1/(2*pi) + x; axis([1 2 3 4]);axis; plot(t,v,'b');
   title('Synthesis of pulse waveform: N terms');
   xlabel('time, t'); ylabel('approximation of v(t)');
   grid;pause;
end;
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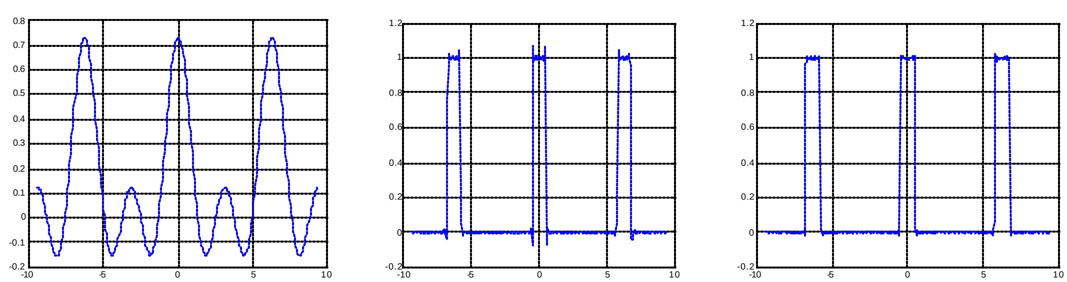
Summation is for N= 2 Terms **N=102 Terms** **N=202 Terms**



Gibbs Phenomenon: Approximating an ideal rectangular pulse (discontinuous at the edges of the rectangle) results always in ripple and overshoot not to exceed 8.95% of the peak amplitude value for large values of terms used in the summation.

Frequency Spectrum for periodic signals: The set of coefficients for periodic signlas are called line spectra or line frequency of x(t). They are generally complex quantities and they represent the signal concentration at fundamental frequency $\pm f_0$ and its harmonics.

Tuncation Error: Let us consider the series expansion of the pulse train as an example of a periodic signal:

$$v(t) = \frac{Aa}{T_0} + \sum_{k=1}^{\infty} \frac{A}{kp} Sin(\frac{kpa}{T_0}).e^{jkw_0 t} = \frac{Aa}{T_0} + \sum_{k=1}^{\infty} \frac{2A}{kp} Sin(\frac{kpa}{T_0}) Cos(kwt + f_k) \qquad (4.16)$$

Since all the coefficient are real in this example we have $f_k = 0 \quad for\ all\ k.$

$$v(t) = \frac{Aa}{T_0} + \sum_{k=1}^{\infty} \frac{A}{kp} Sin(\frac{kpa}{T_0}).e^{jkw_0 t} = \frac{Aa}{T_0} + \sum_{k=1}^{\infty} \frac{2A}{kp} Sin(\frac{kpa}{T_0}) Cos(kw_0 t + f_k)$$
$$= V_0 + \sum_{k=1}^{\infty} 2|V_k|.Cos(kw_0 t) \qquad (4.17)$$

Instead of using infinitely many terms, if we use a finite number of terms as we did in the above example then we have a finite sum in (4.17) and *x(t)* is only an approximation. We will now obtain cost of this approximation. Let us assume we will use only *(2L+1)* harmonics in the sum:

$$s_L(t) = \sum_{k=-L}^{L} S_k.e^{jkw_0 t} \qquad (4.18)$$

The error in using this approximation instead of the ideal *v(t)* is simply:

$$e(t) = v(t) - s_l(t) \qquad (4.19)$$

The mean-square value (MSE) of the error is commonly used as a *Goodness of Fit* in many engineering approximations:

$$MSE = \frac{1}{T_0} \int_{<T_0>} e^2(t)dt = \frac{1}{T_0} \int_{<T_0>} [v(t) - s(t)]^2 dt = \frac{1}{T_0} \int_{<T_0>} [v(t) - \sum_{k=-L}^{L} S_k.e^{jkw_0 t}]^2 dt \qquad (4.20)$$

and the objective is to obtain the minimum value of MSE in (4.20).

Power in periodic signals:

$$P_{avg} = \frac{1}{T_0} \int_{<T_0>} P(t)dt = \frac{1}{T_0} \int_{<T_0>} \frac{v^2(t)}{R} dt \qquad (4.21)$$

where *v(t)* is the voltage across an *R Ohms* resistor. If the input voltage is sinusoid: $v(t) = Cos(w_0 t + q)$ then the power is simply:

$$P_{avg} = \frac{1}{T_0}\int_{<T_0>} \frac{A^2}{R} Cos^2(w_0 t + q)dt = \frac{A^2}{2RT_0}\int_{<T_0>}[1 + Cos(2w_0 t + 2q)]dt = \frac{A^2 t}{2RT_0}\Big|_0^{T_0} + 0$$

For *R=1 Oh*m resistor the average power becomes:

$$P_{avg} = \frac{A^2}{2} Watts \qquad (4.22)$$

In the case of a pulse train of Example 4.2, we have:

$$v(t) = V_0 + 2\sum_{k=1}^{\infty}|V_k|.Cos(kw_0 t + \angle V_k)$$

and the normalized power will be sum of power contributions from each term in the summation:

$$P_{avg} = P_0 + P_1 + P_2 + \cdots + P_k + \cdots$$

$$= V_0^2 + 4\frac{|V_1|^2}{2} + 4\frac{|V_2|^2}{2} + \cdots + 4\frac{|V_k|^2}{2} + \cdots \qquad (4.23)$$

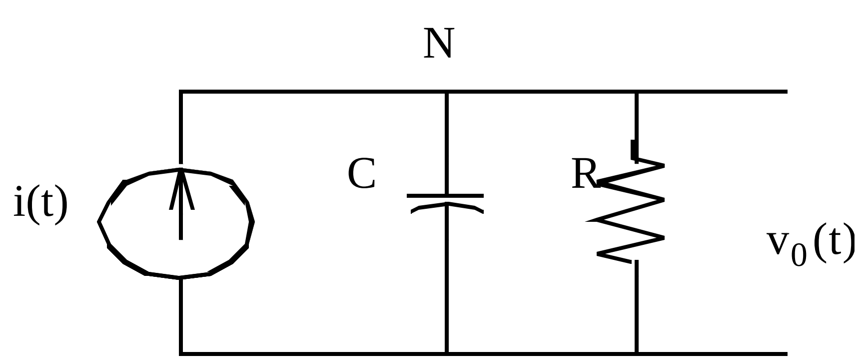
$$= V_0^2 + 2\sum_{k=1}^{\infty}|V_k|^2 = \sum_{k=-\infty}^{\infty}|V_k|^2$$

This is known as the Parseval's Theorem for periodic signals:

$$P_{avg} = \sum_{k=-\infty}^{\infty}|V_k|^2 = \frac{1}{T_0}\int_{<T_0>} P(t)dt \qquad (4.24)$$

where the term in the right is the power in time-domain and the left is the power in frequency-domain and the power is conserved during this Fourier series representation or transformation.

Example 4.3 Find the output voltage $v_0(t)$ from the following RC circuit if the source is an exponential current source: $i(t) = e^{jwt}$.



KCL Equation at Node **N**:

$$i(t) = C.\frac{dv_0(t)}{dt} + \frac{v_0(t)}{R} \qquad (4.25)$$

From the circuit theory knowledge that responses of sinusoids are also sinusoid signals, we have:

$$v(t) = H(jw).e^{jwt} \tag{4.26}$$

We can reformulate the KCL equation of (4.25):

$$\begin{aligned} e^{jwt} &= jwC.H(jw).e^{jwt} + \frac{1}{R}.H(jw).e^{jwt} \\ 1 &= jwC.H(jw) + \frac{1}{R}.H(jw) \end{aligned} \tag{4.27}$$

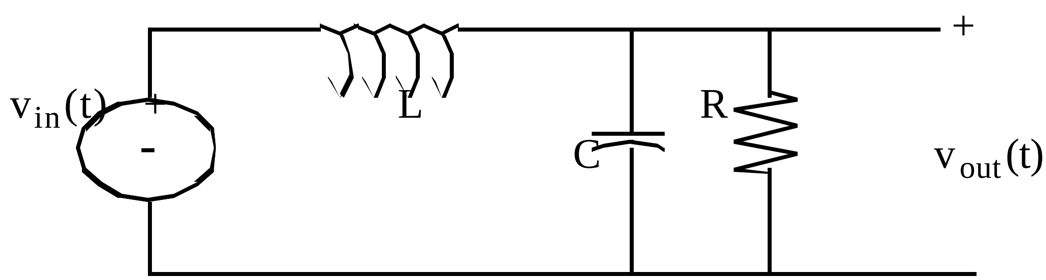
From the last result we have the *system frequency response*:

$$H(jw) = \frac{1}{\frac{1}{R} + jwC} \tag{4.28}$$

The output signal will be simply a substitution of this into (4.26).

$$v_0(t) = \frac{1}{\frac{1}{R} + jwC}.e^{jwt} = \frac{R}{1 + jwRC}.e^{jwt} \tag{4.29}$$

Example 4.4 Find the frequency responses for magnitude and phase of the following system to a sinusoidal voltage $v_{in}(t) = e^{jwt}$.



KVL followed by a voltage division yields:

$$v_{in}(t) = L.\frac{d}{dt}[\frac{v_{out}(t)}{R} + C.\frac{dv_{out}(t)}{dt}] + v_{out}(t) = \frac{L}{R}\frac{dv_{out}(t)}{dt} + LC.\frac{d^2v_{out}(t)}{dt^2} + v_{out}(t)$$

For the given sinusoidal input the response again as in the previous example will be:

$$v_{out}(t) = H(jw).e^{jwt}$$

$$\begin{aligned} e^{jwt} &= \frac{L}{R}.jw.H(jw).e^{jwt} + LC.(jw)^2.H(jw).e^{jwt} + H(jw).e^{jwt} \\ 1 &= \frac{L}{R}.jw.H(jw) + LC.(jw)^2.H(jw) + H(jw) \end{aligned} \tag{4.30}$$

Similarly, the frequency response will be:

$$H(jw) = \frac{1}{1 + jw\frac{L}{R} - w^2 LC} \tag{4.31}$$

The magnitude and the phase (argument) terms are:

$$\left|H(jw)\right| = \frac{1}{\sqrt{(1 - w^2 LC)^2 + (w\frac{L}{C})^2}} \tag{4.32a}$$

$$\angle H(jw) = \arg[H(jw)] = -\arctan\left(\frac{w\frac{L}{R}}{1 - w^2 LC}\right) \tag{4.32b}$$