Chapter 5: z- Transform and Applications

- z-Transform is the discrete-time equivalent of the Laplace transform for continuous signals.
- It is seen as a generalization of the DTFT that is applicable to a very large class of signals observed in diverse engineering applications.

5.1 z-Transform and its Inverse

z-transform: It is a transformation that maps Discrete-time (DT) signal *x*[*n*] into a function of the complex variable z,namely:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (51.)$$

The domain of $X(z)$ or Region of Convergence (ROC) is the set of all z in the complex plane such that the series converges absolutely, i.e.,

$$Dom(X) = \{z \in C; \quad \sum_{n=-\infty}^{\infty} | x[n] z |^{-n} < \infty \qquad (5.2)$$

- Both $X(z)$ and ROC is needed to specify an z-transform.
- ROC depends on $|z|$; $\; if \; z \in ROC$ then $|z|$; $\; if \; ze^{jf} \in ROC$ for any angle f.
- Within ROC, $X(z)$ is an analytic function of the complex variable z, $X(z)$ is a smooth function and derivative exists.

Example 5.1 Find the z-transforms of

$$x[n] = \begin{cases} (1/2)^n & n \geq 0 \\ 0 & n < 0 \end{cases} \quad \text{and} \quad y[n] = \begin{cases} -(1/2)^n & n < 0 \\ 0 & n \geq 0 \end{cases}$$

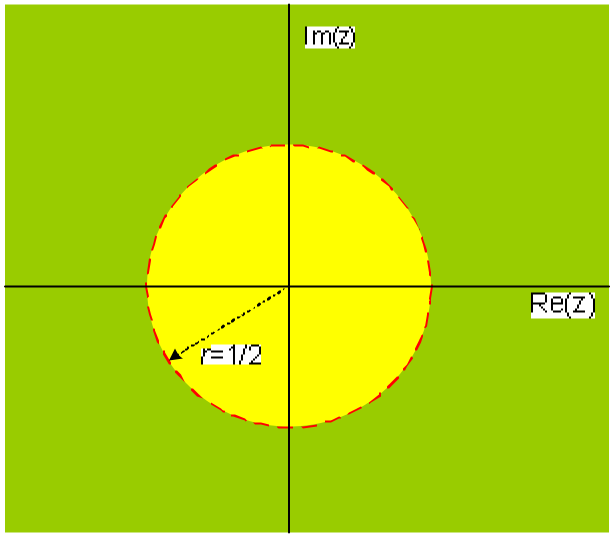
$$X(z)=\sum_{n=0}^{\infty}(1/2)^n.z^{-n}=\sum_{n=0}^{\infty}(\frac{1}{2}z^{-1})^n$$

$$=\frac{1}{1-\frac{1}{2}z^{-1}}=\frac{z}{z-1/2}$$

$$Y(z)=\sum_{n=-\infty}^{-1}[-(1/2)^n.z^{-n}]=-\sum_{n=-\infty}^{-1}(\frac{1}{2}z^{-1})^n$$

$$=-\sum_{m=1}^{\aleph}(2z)^m$$

$$=\frac{2z}{1-2z}=\frac{z}{z-1/2}$$



Both sequences have the same z-transform. The difference is in the two different regions of convergence for (5.1).

- For $X(z)$ the transform will exist if $|\frac{1}{2}z^{-1}|<1$ or $|z|>\frac{1}{2}$; outside the circle.
- For $Y(z)$ the transform will exist if $|2z|<1$ or $|z|<\frac{1}{2}$; inside the circle.
- Appropriate portions of z-plane are called the **ROC**s for $X(z)$ and $Y(z)$, respectively.

Example 5.2 Find the z-transform of

$$x[n]=\begin{cases}3^n & n<0\\(1/3)^n & n=0,2,4,\cdots\\(1/2)^n & n=1,3,5,\cdots\end{cases}$$

$$X(z)=\sum_{n=-\infty}^{-1}3^n.z^{-n}+\sum_{n=0,\ even}^{\infty}(1/3)^n.z^{-n}+\sum_{n=0,\ odd}^{\infty}(1/2)^n.z^{-n}$$

Let $n=-m$, $n=2m$, $n=2m+1$ in the first, second, and third sums, respectively,

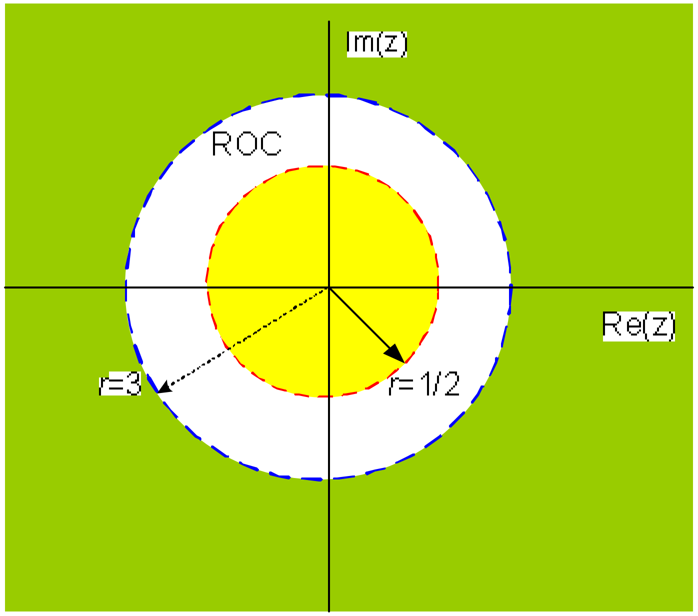
$$X(z)=\sum_{m=1}^{\infty}(\frac{1}{3}z)^{-m}+\sum_{m=0}^{\infty}(\frac{1}{9}z^{-2})^{-m}+\frac{z^{-1}}{2}\sum_{m=0}^{\infty}(\frac{1}{4}z^{-2})^{-m}$$

$$=\sum_{m=0}^{\infty}(\frac{1}{3}z)^{-m}-1+\sum_{m=0}^{\infty}(\frac{1}{9}z^{-2})^{-m}+\frac{z^{-1}}{2}\sum_{m=0}^{\infty}(\frac{1}{4}z^{-2})^{-m}$$

$$=\frac{1}{1-\frac{1}{3}z}-1+\frac{1}{1-\frac{1}{9}z^{-2}}+\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{9}z^{-2}}$$

$$=\frac{1-1+\frac{1}{3}z}{1-\frac{1}{3}z}+\frac{1}{1-\frac{1}{9}z^{-2}}+\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{9}z^{-2}}$$

$$=\frac{\frac{1}{3}z}{1-\frac{1}{3}z}+\frac{1}{1-\frac{1}{9}z^{-2}}+\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{9}z^{-2}}$$



$X(z)$ has poles at $z=\{3,\frac{1}{3},-\frac{1}{3},\frac{1}{2},-\frac{1}{2}\}$. The pole at $z=3$ corresponds to the only anti-causal pole and the other four are causal poles in the ROC of $1/2<|z|<3$.

Inverse z-transform: Let $X(z)$ with an ROC $\Re_X$ be the z-transform a discrete-time signal *x*[*n*] then the inverse transform is defined by:

$$x[n]=\frac{1}{j2p}\oint_C X(z).z^{n-1}dz \qquad (5.3)$$

where *C* is any closed counter-clockwise contour around $z=0$ within $\Re_X$. This integral is evaluated in calculus using the residue theorem. However, this may not be necessary in many cases through power-series expansion. In particular, when $X(z)$ is a rational function:

$$X(z) = \frac{b_0 + b_1 z + \cdots + b_M z^M}{a_0 + a_1 z + \cdots + a_N z^N} \quad \text{with } N \geq M \tag{5.4}$$

Example 5.3 Find the inverse z-transform of

$$X(z) = \frac{z}{z - 0.1}; \quad |z| > 0.1$$

Power series expansion in powers of z^{-1}:

$$\frac{z}{z - 0.1} = 1 + 0.1.z^{-1} + (0.1.z^{-1})^2 + (0.1.z^{-1})^3 + \cdots$$

Note that:

$$x[0] = 1, \quad x[1] = 0.1, \quad x[2] = (0.1)^2, \quad x[3] = (0.1)^3, \cdots$$

which is simply:

$$x[n] = (0.1)^n .u[n]$$

Example 5.4 Find the inverse z-transform of

$$X(z) = \frac{z^3 - z^2 + z - \frac{1}{16}}{z^3 - \frac{5}{4}z^2 + \frac{1}{2}z - \frac{1}{16}}; \quad |z| > 1/2$$

Since the numerator and the denominator have the same power $N = 3$, we write $X(z)$ as the sum of a constant and a remainder form:

$$X(z) = 1 + \frac{\frac{1}{4}z^2 + \frac{1}{2}z}{z^3 - \frac{5}{4}z^2 + \frac{1}{2}z - \frac{1}{16}} = 1 + \frac{\frac{1}{4}z(z+2)}{(z - \frac{1}{2})^2 (z - \frac{1}{4})} = 1 + z[\frac{\frac{1}{4}(z+2)}{(z - \frac{1}{2})^2 (z - \frac{1}{4})}]$$

The bracketed term has a partial fraction expansion:

$$X(z) = 1 + z\left[\frac{-9}{(z-\frac{1}{2})^2} + \frac{5/2}{(z-\frac{1}{2})^2} + \frac{9}{z-\frac{1}{4}}\right] = 1 - 9\frac{z}{z-1/2} + 5\frac{z/2}{(z-1/2)^2} + 9\frac{z}{z-1/4}$$

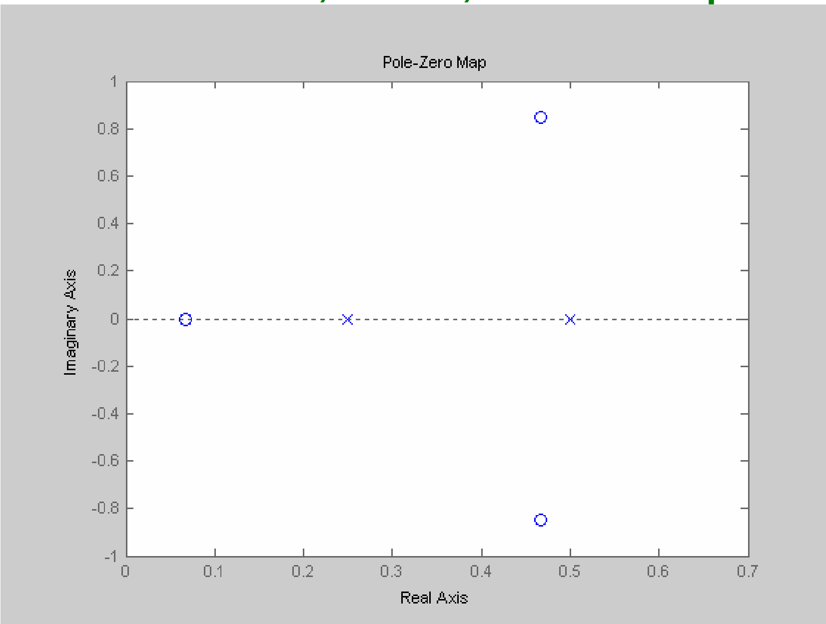
From a standard z-transform tables we find the inverse transform terms as:

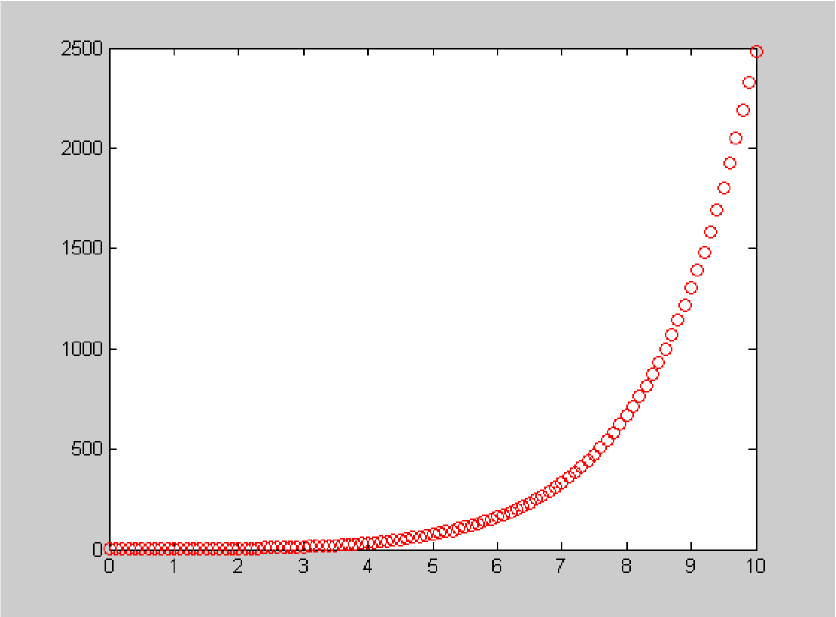
$$x[n] = d[n] - 9(\frac{1}{2})^n.u[n] + 5n(\frac{1}{2})^n.u[n] + 9(\frac{1}{4})^n.u[n]$$

Verification by Matlab:

```
% Using residue function we locate poles p(i), zeros z(i)
% and multiplicative factors k of an X(z) function.
    b=[1,-1,1,-1/16];
    a=[1,-5/4,1/2,-1/16];
    pzmap(b,a);
    [r,p,k]=residue(b,a)
    figure;
% Plot unit-step response
    time=0:0.1:10;
    response=step(b,a,time);
    plot(time,response,'rO'); axis;
```

Results: r = -2.0000, 1.2500, 2.2500 p = 0.5000, 0.5000, 0.2500 k = 1





5.2 Fundamental Properties of z-Transform and Examples

z-Transform tables are very similar to those of Laplace and DTFT transforms and it has several important properties just like them. The critical ones have been tabulated in Table 5.1.

TABLE 5.1 PROPERTIES OF z-TRANSFORM

1. Linearity	$A.x_1[n]+B.x_2[n]$	$A.X_1(z)+B.X_2(z)$
2. Time-Shift (Delay)	$x[n+N]$	$z^{N}[X(z)-\sum_{m=0}^{N-1}x[m].z^{-m}]$
	$x[n-N]$	$z^{-N}[X(z)-\sum_{m=-N}^{-1}x[m].z^{-m}]$
3. Frequency Scaling	$a^{n}.x[n]$	$X(a^{-1}.z)$
4. Multiplication by n	$n.x[n]$	$-z\frac{d}{dz}X(z)$
	$n^{k}.x[n]$	$(-z\frac{d}{dz})^{k}X(z)$
5. Convolution	$x[n]*h[n]$	$X(z).H(z)$

In addition, initial and final values are very useful concepts in explaining systems behavior.

Initial Value: $$x[0]=\lim_{z\to\infty}X(z) \qquad (5.5a)$$

Final Value: $$\lim_{z\to 1}[(1-z^{-1})X(z)]=\lim_{N\to\infty}\sum_{n=0}^{N}(x[n]-x[n-1])=\lim_{N\to\infty}x[N]=x[\infty] \qquad (5.5b)$$

Example 5.5 Find the initial and final values of:

$$X(z)=\frac{z^3-3/4.z^2+2z-5/4}{(z-1)(z-1/3)(z^2-1/2.z+1)}=\frac{z^3}{z^4}.\frac{1-3/4.z^{-1}+2z^{-2}-5/4.z^{-3}}{(1-z^{-1})(1-1/3z^{-1})(1-1/2.z^{-1}+z^{-2})}$$

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} (\frac{z^3}{z^4}).(\frac{1-3/4.z^{-1}+2z^{-2}-5/4.z^{-3}}{(1-z^{-1})(1-1/3z^{-1})(1-1/2.z^{-1}+z^{-2})}) = \lim_{z\to\infty} (\frac{1}{z}).1 = 0$$

$$x[\infty] = \lim_{z\to 1}[(1-z^{-1})X(z)] = \lim_{z\to 1} \frac{z-1}{z} X(z) = \lim_{z\to 1} \frac{z-1}{z} \frac{z^3-3/4.z^2+2z-5/4}{(z-1)(z-1/3)(z^2-1/2.z+1)}$$

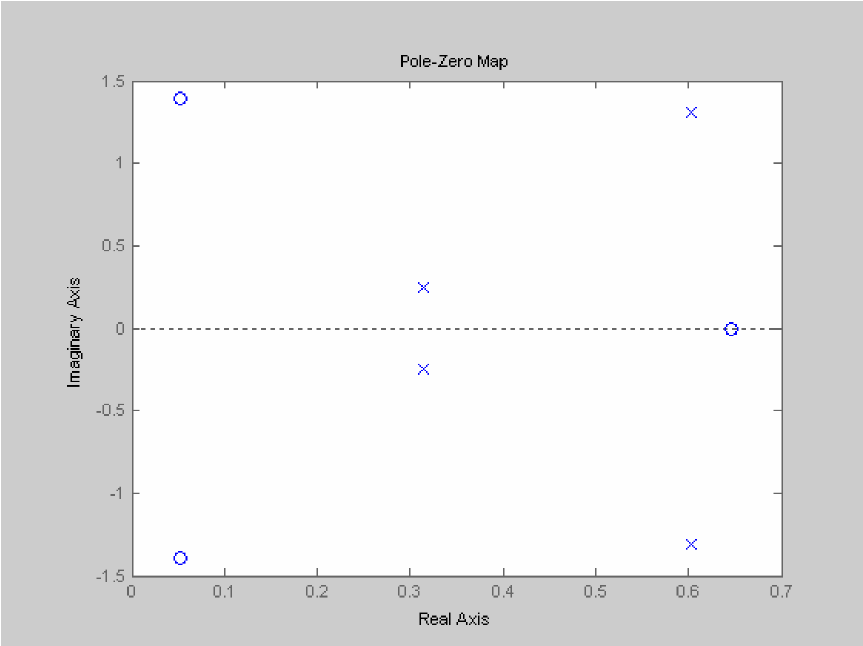
$$= \lim_{z\to 1}[\frac{z^3-3/4.z^2+2z-5/4}{z(z-1/3)(z^2-1/2.z+1)} = \frac{1-3/4+2-5/4}{1.2/3.(2-1/2)} = 1$$

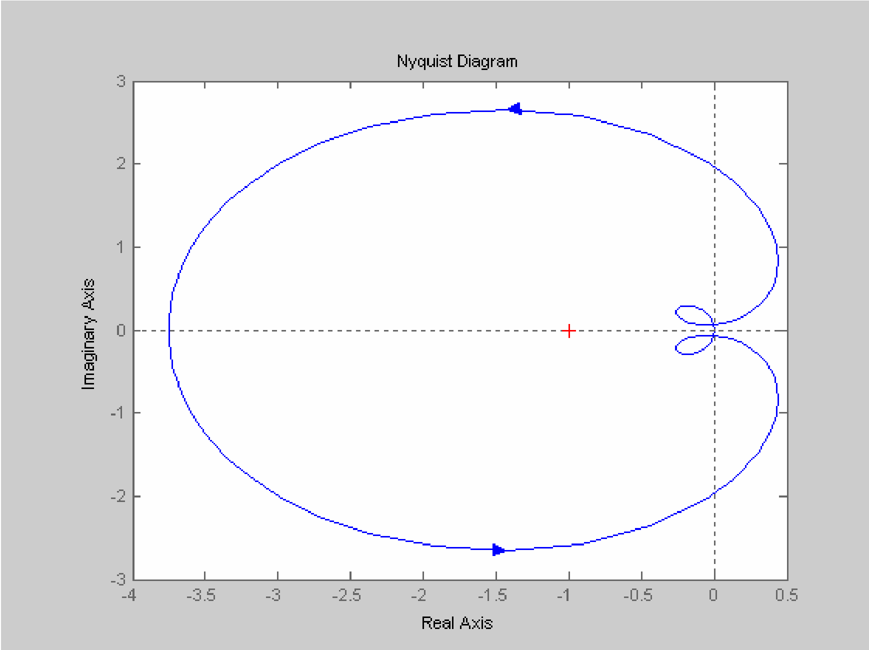
Matlab Analysis:

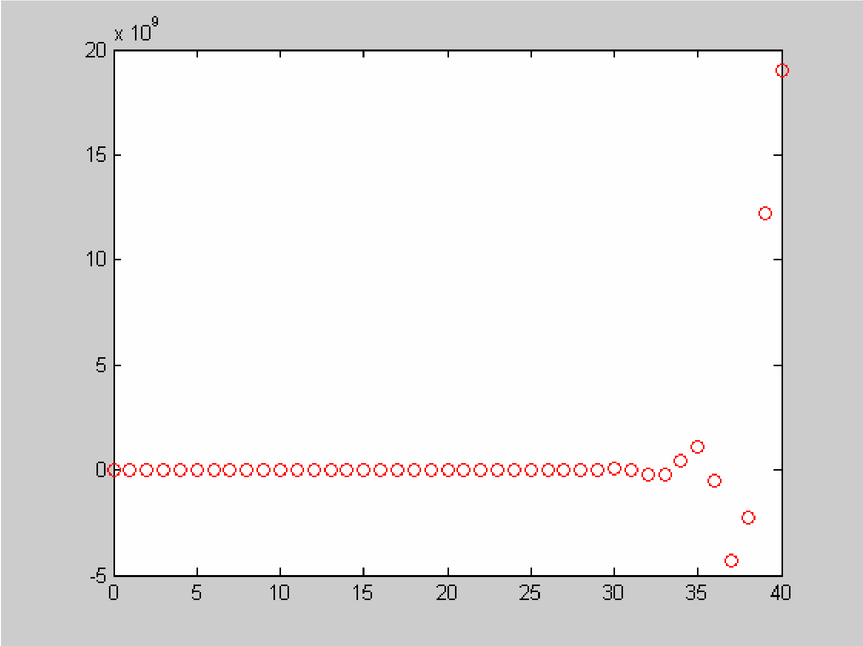
% Matlab Analysis of Example 5.5

```
% Pole-zero map
        b=[1,-3/4,2,-5/4];
        a=[1,-11/6,3,-3/2,1/3];
        pzmap(b,a);
        [r,p,k]=residue(b,a)
        figure;
% Nyquist Analysis
        nyquist(b,a)
        figure
% Plot impulse response
        time=0:1:40;
        response=impulse(b,a,time);
        plot(time,response,'ro');
        axis;
```

r = [0.0615 - 0.4350i, 0.0615 + 0.4350i, 0.4385 + 0.8195i, 0.4385 - 0.8195i]
p = [0.6030 + 1.3114i, 0.6030 - 1.3114i, 0.3137 + 0.2482i, 0.3137 - 0.2482i]
k = [ ]


Pole-Zero Map
Imaginary Axis
Real Axis
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7


Nyquist Diagram
Imaginary Axis
Real Axis
3
2
1
0
-1
-2
-3
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5


x 10^9
20
15
10
5
0
-5
0
5
10
15
20
25
30
35
40

5.8

Example 5.6 Find the z-transform and ROC of

$$x[n]=\cos(w_C n).u[n]=\frac{1}{2}e^{jw_C n}.u[n]+\frac{1}{2}e^{-jw_C n}.u[n]$$

$$X(z)=\frac{1}{2}\sum_{n=0}^{\infty}e^{jw_C n}.z^{-n}+\frac{1}{2}\sum_{n=0}^{\infty}e^{-jw_C n}.z^{-n}=\frac{1}{2}\frac{1}{1-e^{jw_C}.z^{-1}}+\frac{1}{2}\frac{1}{1-e^{-jw_C}.z^{-1}}$$

$$=\frac{1-z^{-1}.\cos w_C}{1-2.z^{-1}.\cos w_C+z^{-2}}$$

Since denominator terms above must satisfy: $|z|>|e^{jw_C}|=|e^{-jw_C}|=1$, ROC: $|z|>1$.

Example 5.7 Find the response from the system (convolution) of:

$x[n]=\{\bar{1},-2a,a^2\}$ and

$h[n]=\{\bar{1},-2a,a^2,a^3,a^4\}$

and a is a complex quantity.

$$X(z)=1-2a.z^{-1}+a^2.z^{-2}=(1-a.z^{-1})^2$$

$$H(z)=1+a.z^{-1}+a^2.z^{-2}+a^3.z^{-3}+a^4.z^{-4}=\frac{1-a^5.z^{-5}}{1-a.z^{-1}}$$

$$y[n]=x[n]*h[n]\Rightarrow \quad Y(z)=X(z).Y(z)=(1-a.z^{-1}).((1-a^5.z^{-5})=1-a.z^{-1}-a^5.z^{-5}+a^5.z^{-6}$$

The inverse z-transform yields the response in discrete time-domain:

$$y[n]=\{\bar{1},-a,0,0,0,-a^5,a^6\}$$

Example 5.8 Similar to Example 5.7 use z-transform to find the convolution of two sequences:

$h[n]=\{1,2,0,-1,1\}$ and $x[n]=\{1,3,-1,-2\}$

$H(z)=1+2z^{-1}-z^{-3}+z^{-4}$ and $X(z)=1+3z^{-1}-z^{-2}-2z^{-3}$

$$Y(z)=X(z).H(z)=1+5z^{-1}+5z^{-2}-5z^{-3}-6z^{-4}+4z^{-5}+z^{-6}-2z^{-7}$$

$$y[n]=\{1,5,5,-5,-6,4,1,-2\}$$

5.3 DTFT and z-Transform Relationship

z-Transform is more general than the DTFT with $z = r.e^{jw}$ with possibility of adjusting r so that the series:

$$Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n].z^{-n} = \sum_{n=-\infty}^{\infty} x[n].r^{-n}.e^{-jwn} = DTFT\{x[n].r^{-n}\} \qquad (5.6)$$

converges.

Example 5.9 Consider $x[n] = 2^n.u[n]$, DTFT does NOT exist since 2^n grows unboundedly as $n \to \infty$. But the series in (5.6) converges provided:

$|2.z^{-1}| < 1$, which stands for a ROC of $|z| > 2$.

$$X(z) = \sum_{n=0}^{\infty} 2^n.z^{-n} = \sum_{n=0}^{\infty} (2.z^{-1})^n = \frac{1}{1-2z^{-1}}$$

If $r = 1$ then $z = e^{jw} \in ROC$ and

$$X(e^{iw}) = \sum_{n=-\infty}^{\infty} x[n].e^{-jwn} = DTFT\{x[n]\} \qquad (5.7)$$

Hence, in many cases, DTFT uses either $X(e^{jw})$ or simply $X(w)$.

5.3 Rational z-Transform

$X(z)$ is a rational in z or (z^{-1}) if we can write it as:

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z + a_2 z^{-2} + \cdots + a_N z^{-N}} \qquad (5.8)$$

or similarly in z^{-1}.

- Rational z-transform plays an important role in DSP, in particular, in studying IIR filters.
- Pole-zero characterizations are critical in determining realizable and stable systems.
- Inversion by means of partial fraction expansion is a practical procedure for causal systems.