Chapter 4: Discrete-time Fourier Transform (DTFT)
4.1 DTFT and its Inverse

Forward DTFT: The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable w, namely:

$$X(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}, \quad w \in \Re \tag{4.1}$$

- Note n is a discrete-time instant, but w represent the continuous real-valued frequency as in the continuous Fourier transform. This is also known as the analysis equation.
- In general $X(w) \in C$
- $X(w + 2np) = X(w) \quad \Rightarrow w \in \{-p, p\}$ is sufficient to describe everything. (4.2)
- $X(w)$ is normally called the spectrum of $x[n]$ with:

$$X(w) = |X(w)|.e^{j\angle X(w)} \quad \Rightarrow \begin{cases} |X(w)|: Magnitude\,Spectrum \\ \angle X(w): Phase\ Spectrum, angle \end{cases} \tag{4.3}$$

- The magnitude spectrum is almost all the time expressed in decibels (dB):

$$|X(w)|_{dB} = 20.\log_{10} |X(w)| \tag{4.4}$$

Inverse DTFT: Let $X(w)$ be the DTFT of $x[n]$. Then its inverse is inverse Fourier integral of $X(w)$ in the interval $\{-p, p)$.

$$x[n] = \frac{1}{2p}\int_{-p}^{p} X(w)e^{jwn}dw \tag{4.5}$$

This is also called the synthesis equation.

Derivation: Utilizing a special integral: $\int_{-p}^{p} e^{jwn}dw = 2pd[n]$ we write:

$$\int_{-p}^{p} X(w)e^{jwn}dw = \int_{-p}^{p} \{ \sum_{k=-\infty}^{\infty} x[k]e^{-jwk} \} e^{jwn} dw = \sum_{k=-\infty}^{\infty} x[k] \int_{-p}^{p} e^{-jw[n-k]} dw = 2p \sum_{k=-\infty}^{\infty} x[k]d[n-k] = 2p . x[n]$$

Note that since x[n] can be recovered uniquely from its DTFT, they form Fourier Pair: $x[n] \Leftrightarrow X(w)$.

Convergence of DTFT: In order DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$ must converge. In other words:

$$X_M(w) = \sum_{n=-M}^{M} x[n]e^{-jwn} \text{ must converge to a limit } X(w) \text{ as } M \to \infty. \quad (4.6)$$

Convergence of $X_m(w)$ for three difference signal types have to be studied:

- Absolutely summable signals: $x[n]$ is absolutely summable iff $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$. In this case, $X(w)$ always exists because:

$$\left| \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} \right| \le \sum_{n=-\infty}^{\infty} |x[n]| . |e^{-jwn}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (4.7)$$

- Energy signals: Remember $x[n]$ is an energy signal iff $E_x \equiv \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$. We can show that $X_M(w)$ converges in the *mean-square* sense to $X(w)$:

$$\lim_{M \to \infty} \int_{-p}^{p} |X(w) - X_M(w)|^2 dw = 0 \quad (4.8)$$

Note that mean-square sense convergence is weaker than the uniform (always) convergence of (4.7).

- Power signals: $x[n]$ is a power signal iff

$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty$$

- In this case, $x[n]$ with a finite power is expected to have infinite energy. But $X_M(w)$ may still converge to $X(w)$ and have DTFT.
- Examples with DTFT are: periodic signals and unit step-functions.
- $X(w)$ typically contains continuous delta functions in the variable w.

4.2 DTFT Examples

Example 4.1 Find the DTFT of a unit-sample $x[n] = d[n]$.

$$X(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} d[n]e^{-jwn} = e^{-j0} = 1 \qquad (4.9)$$

Similarly, the DTFT of a generic unit-sample is given by:

$$DTFT\{d[n-n_0]\} = \sum_{n=-\infty}^{\infty} d[n-n_0]e^{-jwn} = e^{-jwn_0} \qquad (4.10)$$

Example 4.2 Find the DTFT of an arbitrary finite duration discrete pulse signal in the interval: $N_1 < N_2$:

$$x[n] = \sum_{k=-N_1}^{N_2} c_k d[n-k]$$

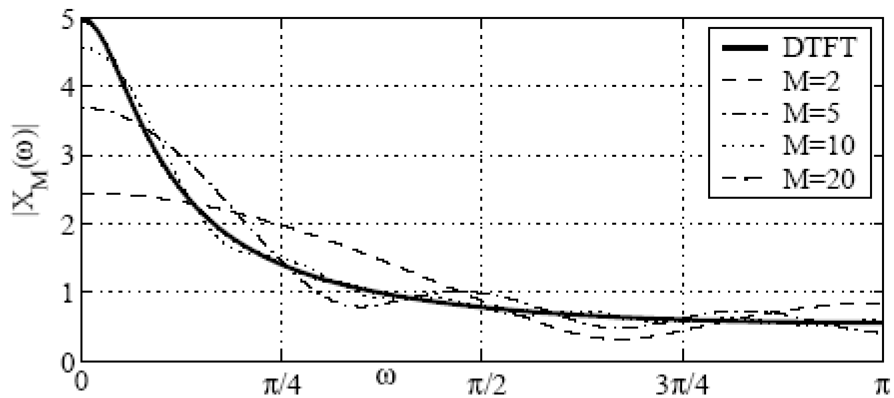
Note: $x[n]$ is absolutely summable and DTFT exists:

$$X(w) = \sum_{n=-\infty}^{\infty} \{ \sum_{k=-N_1}^{N_2} c_k d[n-k]\}e^{-jwn} = \sum_{k=-N_1}^{N_2} c_k \{ \sum_{n=-\infty}^{\infty} d[n-k]e^{-jwn}\} = \sum_{k=-N_1}^{N_2} c_k e^{-jwk} \qquad (4.11)$$

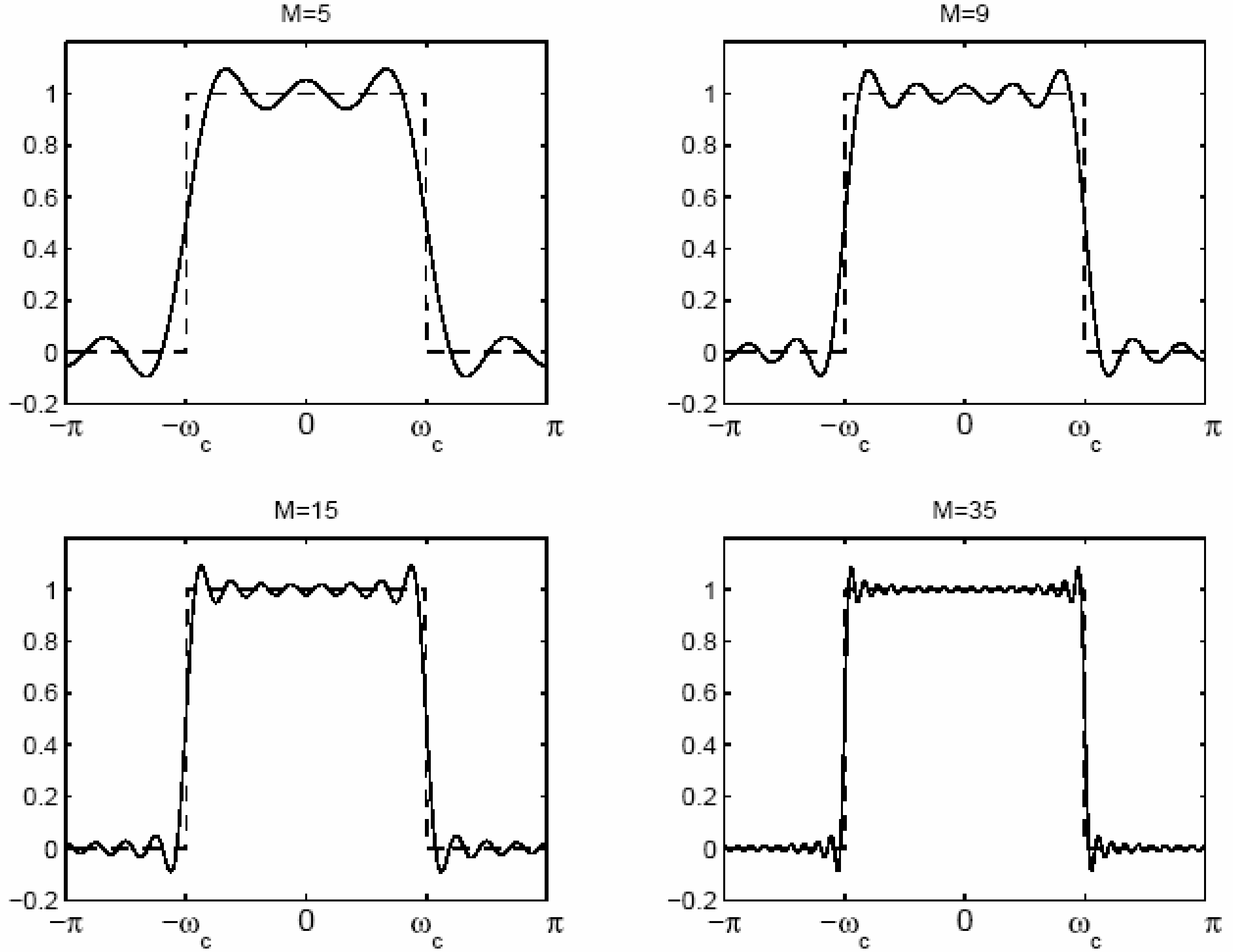
Example 4.3 Find the DTFT of an exponential sequence: $x[n] = a^n u[n] \quad where\ |a| < 1$. It is not difficult to see that this signal is absolutely summable and the DTFT must exist.

$$X(w) = \sum_{n=-\infty}^{\infty} a^n .u[n]e^{-jwn} = \sum_{n=0}^{\infty} a^n .e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1-ae^{-jw}} \qquad (4.12)$$

Observe the plot of the magnitude spectrum for DTFT and $X_M(w)$ for: $a = 0.8$ and $M = \{2,5,10,20,\infty = DTFT\}$



Example 4.4 Gibbs Phenomenon: Significance of the finite size of M in (4.6).

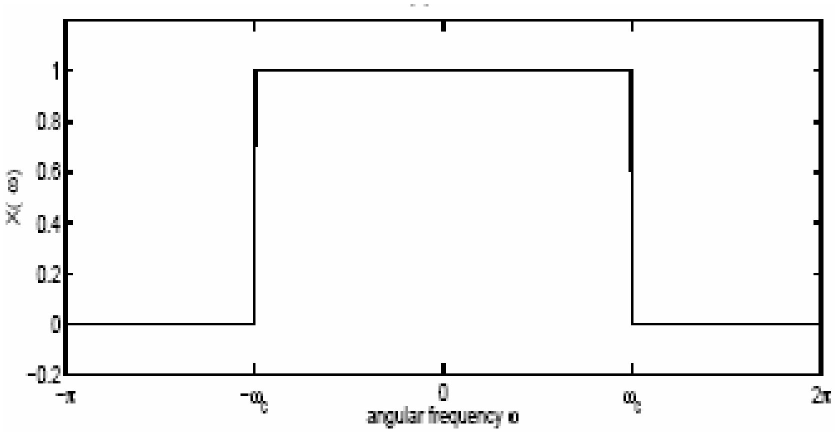


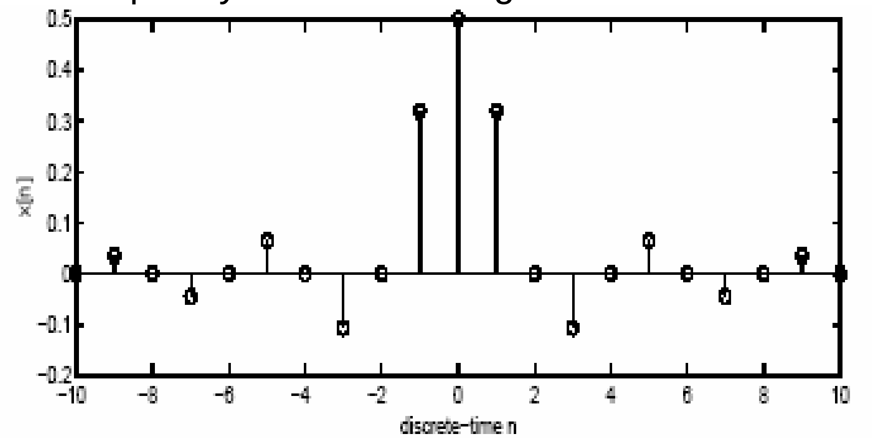
For small M, the approximation of a pulse by a finite harmonics have significant overshoots and undershoots. But it gets smaller as the number of terms in the summation increases.

Example 4.5 Ideal Low-Pass Filter (LPF). Consider a frequency response defined by a DTFT with a form:

$$X(w) = \begin{cases} 1 & |w| < w_C \\ 0 & w_C < w < p \end{cases} \tag{4.13}$$

Here any signal with frequency components smaller than w_C will be untouched, whereas all other frequencies will be forced to zero. Hence, a discrete-time continuous frequency ideal LPF configuration.





Through the computation of inverse DTFT we obtain:

$$x[n] = \frac{1}{2\pi} \int_{-w_C}^{w_C} e^{jwn} dw = \frac{w_C}{\pi} Sinc(\frac{w_C n}{\pi}) \qquad (4.14)$$

where $Sinc(x) = \frac{\sin(\pi x)}{\pi x}$. The spectrum and its inverse transform for $w_C = \pi/2$ has been depicted above.

4.3 Properties of DTFT

4.3.1 Real and Imaginary Parts:

$$x[n] = x_R[n] + jx_I[n] \quad \Leftrightarrow \quad X(w) = X_R(w) + jX_I(w) \qquad (4.15)$$

4.3.2 Even and Odd Parts:

$$x[n] = x_{ev}[n] + x_{odd}[n] \quad \Leftrightarrow \quad X(w) = X_{ev}(w) + X_{odd}(w) \qquad (4.16a)$$

$$x_{ev}[n] = 1/2.\{x[n] + x^*[-n]\} = x_{ev}^*[-n] \quad \Leftrightarrow \quad X_{ev}(w) = 1/2.\{X(w) + X^*[-w]\} = X_{ev}^*[-w] \qquad (4.16b)$$

$$x_{odd}[n] = 1/2.\{x[n] - x^*[-n]\} = -x_{odd}^*[-n] \quad \Leftrightarrow \quad X_{odd}(w) = 1/2.\{X(w) - X^*(-w)\} = -X_{odd}^*(-w) \qquad (4.16c)$$

4.3.3 Real and Imaginary Signals:

If $x[n] \in \Re$ then $X(w) = X^*(-)$; even symmetry and it implies:

$$|X(w)|=|X(-w)|; \quad \angle X(w)=-\angle X(-w) \tag{4.17a}$$

$$X_R(w)=X_R(-w); \quad X_I(w)=-X_I(-w) \tag{4.17b}$$

If $x[n]\in \Im$ (purely imaginary) then $X(w)=-X^*(-w)$; odd symmetry (anti-symmetry.)

4.3.4 Linearity:

a. Zero-in zero-out and

b. Superposition principle applies: $A.x[n]+B.y[n] \quad \Leftrightarrow \quad A.X(w)+B.X(w)$ (4.18)

4.3.5 Time-Shift (Delay) Property:

$$x[n-D] \quad \Leftrightarrow \quad e^{-jwD}.X(w) \tag{4.19}$$

4.3.6 Frequency-Shift (Modulation) Property:

$$X[w-w_C] \quad \Leftrightarrow \quad e^{-jw_C n}.x[n] \tag{4.20}$$

Example 4.6 Consider a first-order system:

$$y[n]=K_0.x[n]+K_1.x[n-1]$$

Then $Y(w)=(K_0+K_1.e^{-jw})X(w)$ and the frequency response:

$$H(jw)=Y(w)/X(w)=K_0+K_1.e^{-jw}$$

4.3.7 Convolution Property:

$$x[n]*h[n] \quad \Leftrightarrow \quad X(w).H(w) \tag{4.21}$$

4.3.8 Multiplication Property:

$$x[n].y[n] \quad \Leftrightarrow \quad \frac{1}{2p}\int_{-p}^{p} X(f).Y(w-f)df \tag{4.22}$$

4.3.9 Differentiation in Frequency:

$$j.\frac{dX(w)}{dw} \Leftrightarrow n.x[n] \tag{4.23}$$

4.3.10 Parseval's and Plancherel's Theorems:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2p} \int_{-p}^{p} |X(w)|^2 dw \qquad (4.24)$$

If $x[n]$ and/or $y[n]$ complex then

$$\sum_{n=-\infty}^{\infty} x[n].y^*[n] = \frac{1}{2p} \int_{-p}^{p} X(w).Y^*(w) dw \qquad (4.25)$$

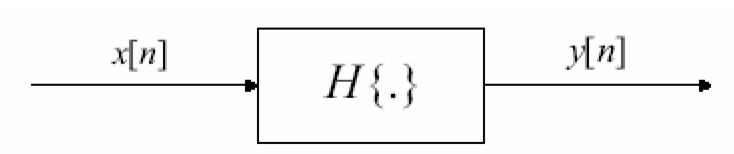
Example 4.7 Find the DTFT of a generic discrete-time periodic sequence $x[n]$.
Let us write the Fourier series expansion of a generic periodic signal:

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jkw_0 n} \quad \text{where } w_0 = \frac{2p}{N}$$

$$X(w) = DTFT\{x[n]\} = DTFT\{\sum_{k=0}^{N-1} a_k e^{jkw_0 n}\} = \sum_{k=0}^{N-1} a_k .DTFT\{e^{jkw_0 n}\} = \sum_{k=0}^{N-1} a_k .2pd(w - kw_0) \qquad (4.26)$$

Therefore, DTFT of a periodic sequence is a set of delta functions placed at multiples of kw_0 with heights a_k.

4.4 DTFT Analysis of Discrete LTI Systems



The input-output relationship of an LTI system is governed by a convolution process:

$y[n] = x[n] * h[n]$ where $h[n]$ is the discrete time impulse response of the system.

Then the frequency-response is simply the DTFT of $h[n]$:

$$H(w) = \sum_{n=-\infty}^{\infty} h[n].e^{-jwn}, \quad w \in \Re \qquad (4.27)$$

- If the LTI system is stable then $h[n]$ must be absolutely summable and DTFT exists and is continuous.
- We can recover $h[n]$ from the inverse DTFT:

$$h[n] = IDTFT\{H(w)\} = \frac{1}{2p}\int_{-p}^{p} H(w).e^{jwn}\,dw \qquad (4.28)$$

- We call $|H(w)|$ as the magnitude response and $\angle H(w)$ the phase response

Example 4.8 Let

$$h[n] = (\frac{1}{2})^n.u[n] \text{ and } x[n] = (\frac{1}{3})^n.u[n]$$

Let us find the output from this system.

1. Via Convolution:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} (\frac{1}{3})^k.u[k].(\frac{1}{2})^{n-k}.u[n-k] \quad \Rightarrow \text{ Not so easy.}$$

2. Via Fast Convolution or DFTF from Example 4.3 or Equation(4.12):

$$H(w) = \frac{1}{1-\frac{1}{2}e^{-jw}} \quad \text{and} \quad X(w) = \frac{1}{1-\frac{1}{3}e^{-jw}}$$

$$Y(w) = X(w).H(w) = \frac{1}{(1-\frac{1}{3}e^{-jw}).(1-\frac{1}{2}e^{-jw})} = \frac{3}{1-\frac{1}{2}e^{-jw}} - \frac{2}{1-\frac{1}{3}e^{-jw}}$$

and the inverse DTFT will result in:

$$y[n] = 3(\frac{1}{2})^n.u[n] - 2(\frac{1}{3})^n.u[n]$$

Example 4.9 Causal moving average system:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

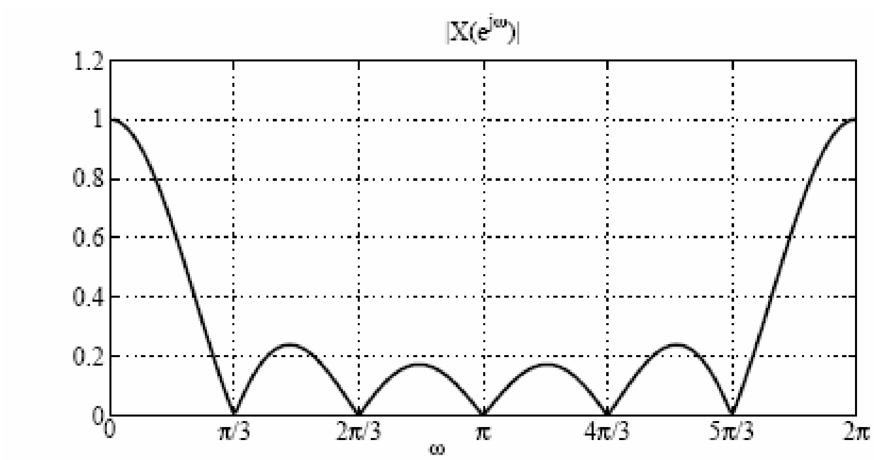
If the input were a unit-impulse: $x[n] = d[n]$ then the output would be the discrete-time impulse response:

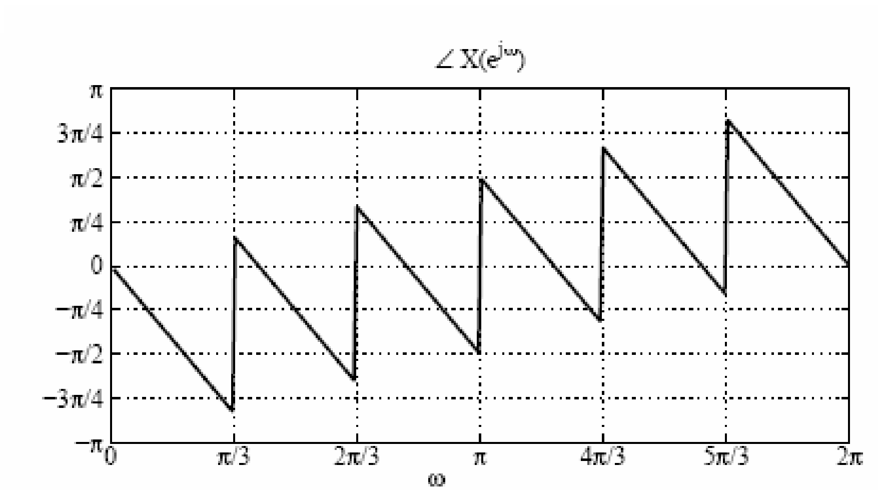
$$h[n]=\frac{1}{M}\sum_{k=0}^{M-1}d[n-k]=\begin{cases}1/M & 0\le n<M\\ 0 & Otherwise\end{cases}=\frac{1}{M}(u[n]-n[n-M])$$

The frequency response:

$$H(w)=\frac{1}{M}\sum_{n=0}^{M-1}e^{-jwn}=\frac{1}{M}\frac{e^{-jwM}-1}{e^{-jw}-1}=\frac{1}{M}\frac{e^{-jwM/2}}{e^{-jw/2}}\frac{e^{-jwM/2}-e^{jwM/2}}{e^{-jw/2}-e^{jw/2}}=\frac{1}{M}.e^{-jw(M-1)/2}.\frac{\sin(wM/2)}{\sin(w/2)}$$

For M=6 we plot the magnitude and the phase response of this system:





Notes:

1. Magnitude response Zeros at $w=\frac{2pk}{M}$ $where$ $\frac{\sin(wM/2)}{\sin(w/w)}=0$
2. Level of first sidelobe $\approx -13\,dB$
3. Phase response with a negative slope of – $(M-1)/2$
4. Jumps of p at $w=\frac{2pk}{M}$ $where$ $\frac{\sin(wM/2)}{\sin(w/w)}$ changes its sign.

TABLE: 4.1 DISCRETE-TIME FOURIER TRANSFORM PAIRS

Signal	DTFT
$d[n]$	1
1	$2p.d(w)$
$e^{jw_C n}$	$2p.d(w - w_C)$
$\sum_{k=0}^{N-1} a_k.e^{jkw_C n} \quad with \quad Nw_C = 2p$	$\sum_{k=0}^{N-1} 2p.a_k.d(w - kw_C)$
$a^n.u[n]; \quad \|n\| < 1$	$\frac{1}{1 - a.e^{-jw}}$
$a^{\|n\|}.; \quad \|n\| < 1$	$\frac{1 - a^2}{1 - 2a.\cos w + a^2}$
$n.a^n.u[n]; \quad \|n\| < 1$	$\frac{a.e^{-jw}}{(1 - a.e^{-jw})^2}$
$rect[\frac{n}{N}]$	$\frac{\sin[w(N + 1/2)]}{\sin[w/2]}$
$\frac{\sin w_C n}{pn}$	$rect[w/2w_C]$

TABLE 4.2 PROPERTIES OF DTFT		
1. Linearity	$A.x_1[n]+B.x_2[n]$	$A.X_1(w)+B.X_2(w)$
2. Time-Shift (Delay)	$x[n-N]$	$e^{-jwN}.X(w)$
3. Frequency-Shift	$x[n].e^{jw_C n}$	$X(w-w_C)$
4. Linear Convolution	$x[n]*h[n]$	$X(w).H(w)$
5. Modulation	$x[n].p[n]$	$\frac{1}{2p}\int_{<2p>} X(h).H(w-h)dh$
6. Periodic Signals	$x[n]=x[n+N]$	$\sum_{k=-\infty}^{\infty} 2p.a_k.d(w-kw_C)$
	$w_C=\frac{2p}{N}$	$a_k=\frac{1}{N}\sum_{<N>} x[n].e^{-jkw_C n}$

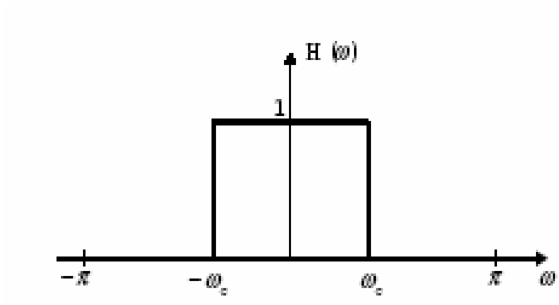
Example 4.10 Response of an LTI system with $H(w)=DTFT\{h[n]\}$: Given $x[n]=e^{jwn}$; a complex harmonic.

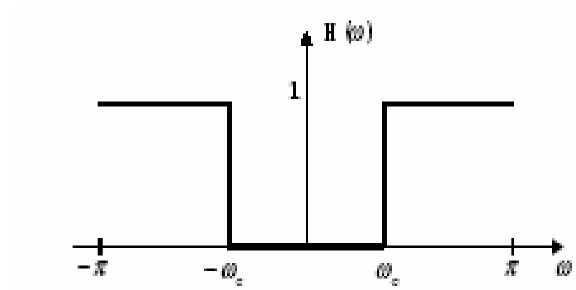
$$y[n]=\sum_k h[k].x[n-k]=\sum_k h[k].e^{jw(n-k)}=\{\sum_k h[k].e^{-jwk}\}.e^{jwn}=H(w).x[n] \quad (4.29)$$

Note that this is the ONLY time frequency-domain variable w and the time-domain variable n appear on the same side of the equation. In all other cases, we have time domain variable in the time-domain and vice versa. In calculus jargon, e^{jwn} acts as the Eigenvector of the system.

4.5 FREQUENCY-SELECTIVE DISCRETE-TIME FILTERS

4.5.1 Ideal Low-Pass and High-Pass Filters

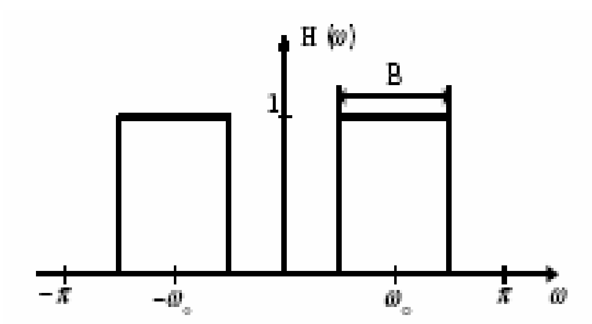


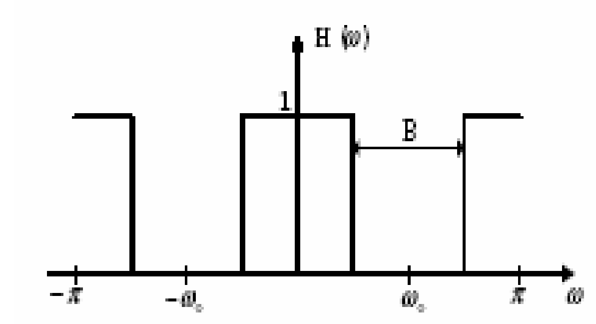


$$H_{LP}(w)=\begin{cases}1 & if\ |w|<w_C\\ 0 & if\ w_C<|w|<p\end{cases} \qquad H_{HP}(w)=\begin{cases}0 & if\ |w|<w_C\\ 1 & if\ w_C<|w|<p\end{cases} \qquad (4.30a\text{-}d)$$

$$h_{LP}[n]=\frac{w_C}{p}.Sinc(\frac{w_C n}{p}) \qquad h_{HP}[n]=d[n]-h_{LP}[n]=d[n]-\frac{w_C}{p}.Sinc(\frac{w_C n}{p})$$

4.5.2 Ideal Band-Pass and Band-Stop Filters





$$H_{BP}(w)=\begin{cases}1 & if\ |w-w_C|<B/2\\ 0 & elsewhere\ in\ (-p,p)\end{cases} \qquad H_{BS}(w)=\begin{cases}0 & if\ |w-w_C|<B/2\\ 1 & elsewhere\ in\ (-p,p)\end{cases} \qquad (4.31a\text{-}d)$$

$$h_{BP}[n]=2.\cos(w_C n).h_{LP}[n]\big|_{w_C=B/2} \qquad h_{BS}[n]=d[n]-h_{BP}[n]=d[n]-2.\cos(w_C n).h_{LP}[n]\big|_{w_C=B/2}$$

- All of these ideal filters are non-causal and hence, non-realizable.
- They form benchmark for implementable real-life filters.

4.6 Phase delay and group delay

Consider an integer system, for which the input-output relationship is given by:

$$y[n] = x[n-k], \quad k \in Integer \tag{4.32a}$$

The frequency response is computed:

$$Y(w) = e^{-jwk}.X(w) \quad \Rightarrow \quad H(w) \equiv \frac{Y(w)}{X(w)} = e^{-jwk} \tag{4.32b}$$

The phase response of this system:

$$\angle H(w) = -wk \tag{4.33}$$

is a linear function of the frequency variable w.

Phase delay t_{PH} is defined by:

$$t_{PH} \equiv -\frac{\angle H(w)}{w} \tag{4.34}$$

For integer systems, this simplifies to:

$$-\frac{\angle H(w)}{w} = k$$

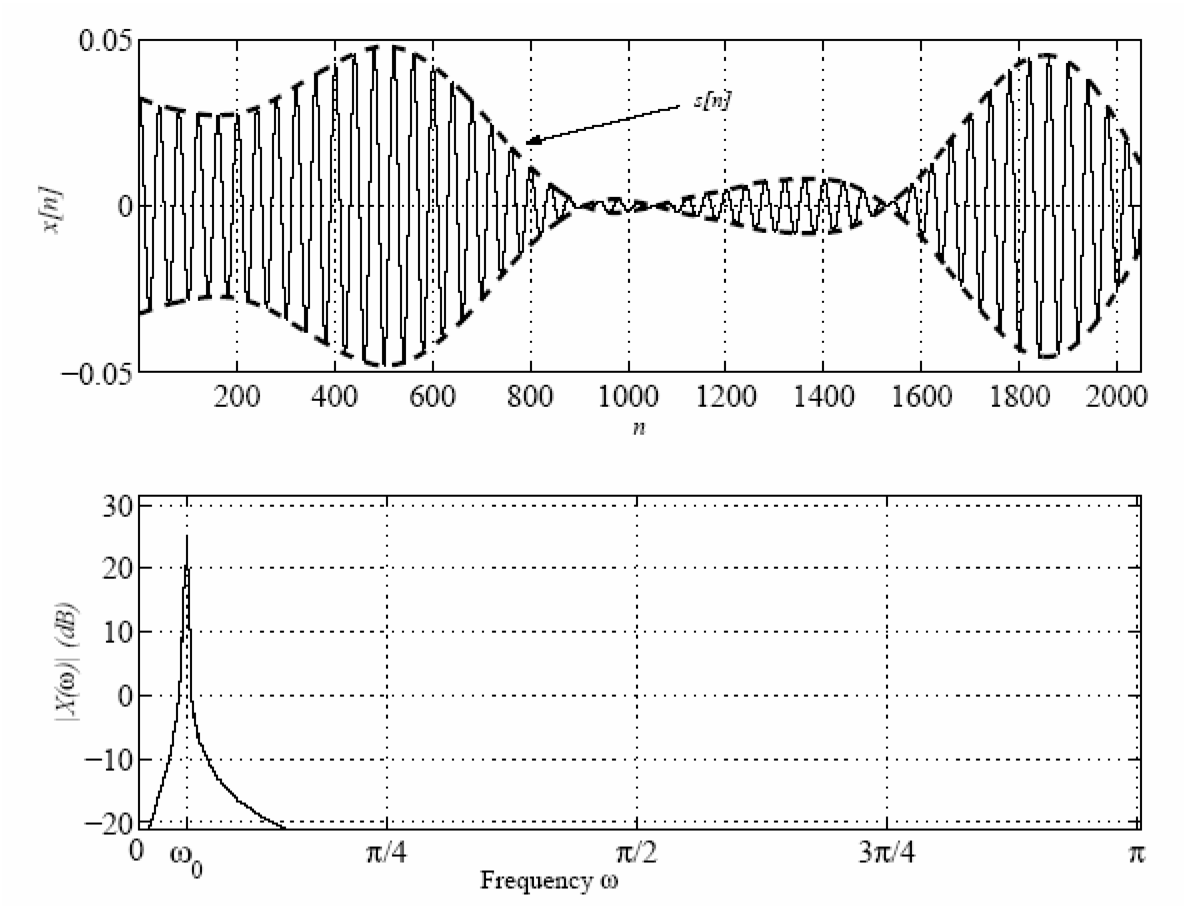
Group Delay is more meaningful and defined by:

$$t_G \equiv -\frac{d\angle H(w)}{dw} \tag{4.35}$$

It is useful for dealing with narrow-band input signal $x[n]$ is centered around a carrier frequency w_0.

$$x[n] = s[n].e^{jw_0 n}$$

where $s[n]$ is a slowly-varying envelope. Typical digital communication task, as shown below.



The corresponding system output is approximated by:

$$y[n] \approx |H(w_0)|.s[n - t_G(w)].e^{jw_0[n - t_{PH}(w_0)]} \qquad (4.36)$$

It is easy to see that the phase delay t_{PH} contributes a phase shift to the carrier $e^{jw_0 n}$, whereas the group delay t_G causes a delay to the envelope $s[n]$.

Pure delay, or All-Pass Filter:
When a system is a pure delay; i.e., its magnitude response is unity for all w and the phase is a linear function of the delay t .

$$|H(w)|=1 \quad t_{PH}(w)=t_{G}(w)=1 \tag{4.37}$$

If the phase is linear but the magnitude may depend on w, then the system is labeled as a linear **Phase system**:

$$H(w)=|H(w)|.e^{j\angle H(w)} \tag{4.38a}$$

where

$$\angle H(w)=-wt \tag{4.38b}$$

where the phase is a linear function of w with a slope $-t$.