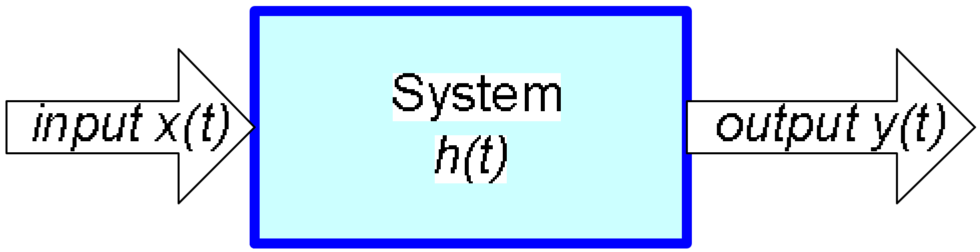
Chapter 3: LINEAR TIME-INVARIANT SYSTEMS
3.1 MOTIVATION

Continuous and discrete-time systems that are both linear and time-in variant (LTI) play a central role in digital signal processing, communication engineering and control applications:

- Many physical systems are either LTI or approximately so.
- Many efficient tools are available for the analysis and design of LTI systems (e.g. spectral analysis).

Consider the general input-output block diagram of a system. The response of the system $h(t)$ to an input signal $x(t)$ is found by a convolution process, which takes into consideration the complete history of the signal and the information in the system memory.



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(u).h(t-u)du \qquad (3.1a)$$

Similarly, for the discrete-time case:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k].h[n-k] \qquad (3.1b)$$

Note that $h[n]$ is called both impulse response and unit pulse response.

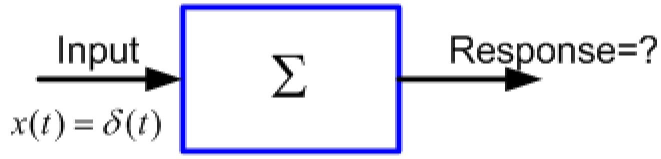
Why impulse response or unit pulse response?

Let the input to the continuous LTI system be $x(t) = d(t)$. Then from the definition of the convolution operation we write:

$$y(t) = x(t) * h(t) = d(t) * h(t) = \int_{-\infty}^{\infty} d(t).h(t-t)dt = h(t) \qquad (3.2)$$

The last integral above is obtained from the ***Sifting Theorem*** definition of the delta function of Chapter 2.

Example 3.1: Impulse response of an accumulator



$$h[n]=\sum_{k=-\infty}^{n}\delta[k]=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

3.2 Properties and Examples of Linear Convolution Process

3.2.1 Commutativity Property: Convolution is a commutative operation, i.e., the roles of $x(t)$ and $h(t)$ can be interchanged. Similarly, for $x[n]$ and $h[n]$.

$$y(t)=x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{\infty}x(t).h(t-t)dt=\int_{-\infty}^{\infty}h(t).x(t-t)dt \tag{3.3a}$$

$$y[n]=x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}x[k].h[n-k]=\sum_{k=-\infty}^{\infty}h[k].x[n-k] \tag{3.3b}$$

3.2.2 Associativity Property: This property will form the basis for cascade (series) systems:

$$y(t)=[x(t)*h_1(t)]*h_2(t)=x(t)*[h_1(t)*h_2(t)]=x(t)*h_S(t) \tag{3.4}$$

where $h_S(t)$ represents the cascade connection of two subsystems $h_1(t)$ and $h_2(t)$, respectively:

$$h_S(t)=h_1(t)*h_2(t) \tag{3.5}$$

Combined cascade system impulse response $h_S(t)$ is equal to the convolution of system responses of individual subsystems. This result can easily be extended to the series connection of many systems via repeated applications of the associativity property.

3.2.3 Distributivity Property: This property, on the other hand, forms the basis for parallel systems.

$$y(t)=[x(t)*h_1(t)]+[x(t)*h_2(t)]=x(t)*[h_1(t)+h_2(t)]=x(t)*h_P(t) \qquad (3.6)$$

As in the previous case, $h_P(t)$ corresponds to the parallel combination of two subsystems.

3.2.4 Linear Convolution Examples (Elementary):

Example 3.2: Convolution of signals with delta and unit-step functions.

$$x(t)*d(t)=\int_{-\infty}^{\infty} x(t).d(t-t)dt = x(t) \qquad x(t)*d(t-t_0)=\int_{-\infty}^{\infty} d(t-t_0).x(t-t)dt = x(t-t_0) \qquad (3.7)$$

$$x(t)*u(t)=\int_{-\infty}^{\infty} x(t).u(t-t)dt = \int_{-\infty}^{t} x(t)dt \qquad (3.8)$$

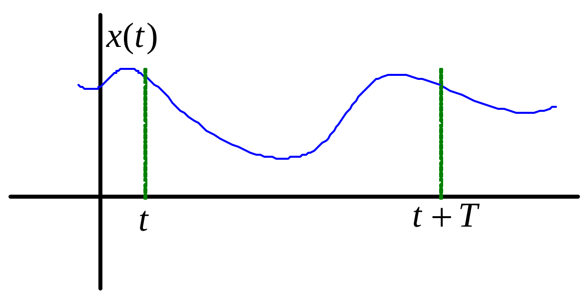
Observation:

- The convolution of any function by a delta function gives the original function and the convolution of any function with a shifted version of the delta function results in the shifted replica of the original function.
- Convolving a signal by a unit-step function is equivalent to a perfect integrator.

Example 3.3: Time Averaging

Time-averaging is frequently employed in finding average behavior or mean of systems or signals or data.

$$\overline{x(t)} = x_{ave}(t)=\frac{1}{T}\int_{t}^{t+T} x(t)dt \qquad (3.9)$$



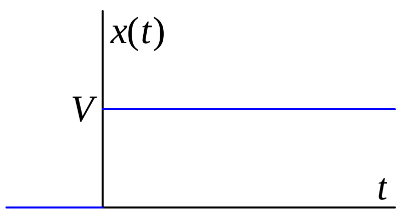
This can be computed in terms of step functions as follows:

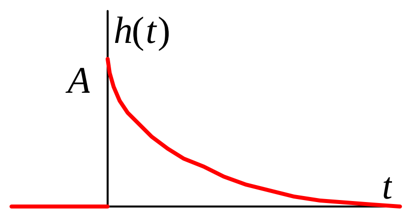
$$\overline{x(t)} = \frac{1}{T}\left[\int_{-\infty}^{t+T} x(t)\,dt - \int_{-\infty}^{t} x(t)\,dt\right] = \frac{1}{T}[x(t) * u(t+T) - x(t) * u(t)]$$
$$= \frac{1}{T} x(t) * [u(t+T) - u(t)] \qquad (3.10)$$

Example 3.4: Response of a Capacitive Circuit to a switched DC voltage, where input and system impulses are simply:

$$x(t) = V.u(t) \quad and \quad h(t) = A.e^{-at}.u(t) \quad where\ a > 0 \qquad (3.11)$$

The task is to compute: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t)h(t-t)\,dt$



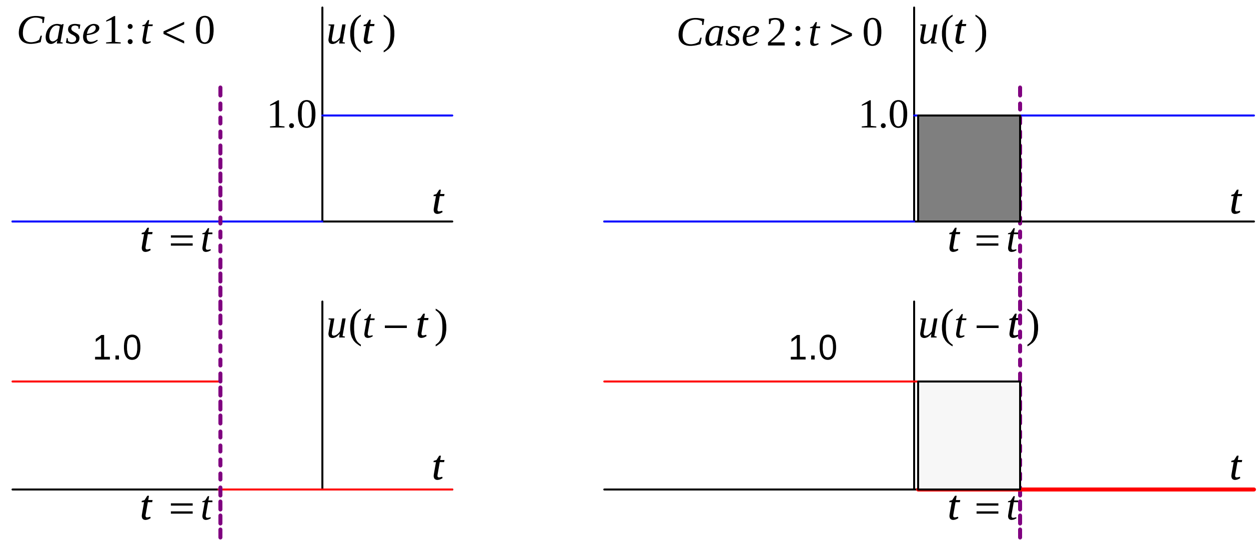


$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-t)\,dt = \int_{-\infty}^{\infty} VAe^{-a(t-t)}u(t)u(t-t)\,dt \qquad (3.12a)$$

Limits of integration are very critical and decided by the non-zero segments of the product of two step functions:

$$u(t).u(t-t). \qquad (3.12b)$$

Let us now find these non-zero segments with graphical support:



Case 1: t<0

Since non-zero segments of the product is zero as clearly seen from the plots, the integral in (3.12a) is also zero to yield:

$$y(t) = AV \int_{-\infty}^{\infty} 0.dt = 0 \qquad (3.13a)$$

Case 2: t>0

Since non-zero segments of the product is the region between *0* and t as shown above, this time the limits of integral this time becomes:

$$y(t) = AV.\int_{0}^{t} e^{-a(t-t)}dt = AV.e^{-at}.\int_{0}^{t} e^{at}\,dt = \frac{AV}{a}.e^{-at}.e^{at}\Big|_{0}^{t} = \frac{AV}{a}.(1-e^{-at}) \qquad (3.13b)$$

When we combine these two results into a single equation using a unit step function we have the final answer:

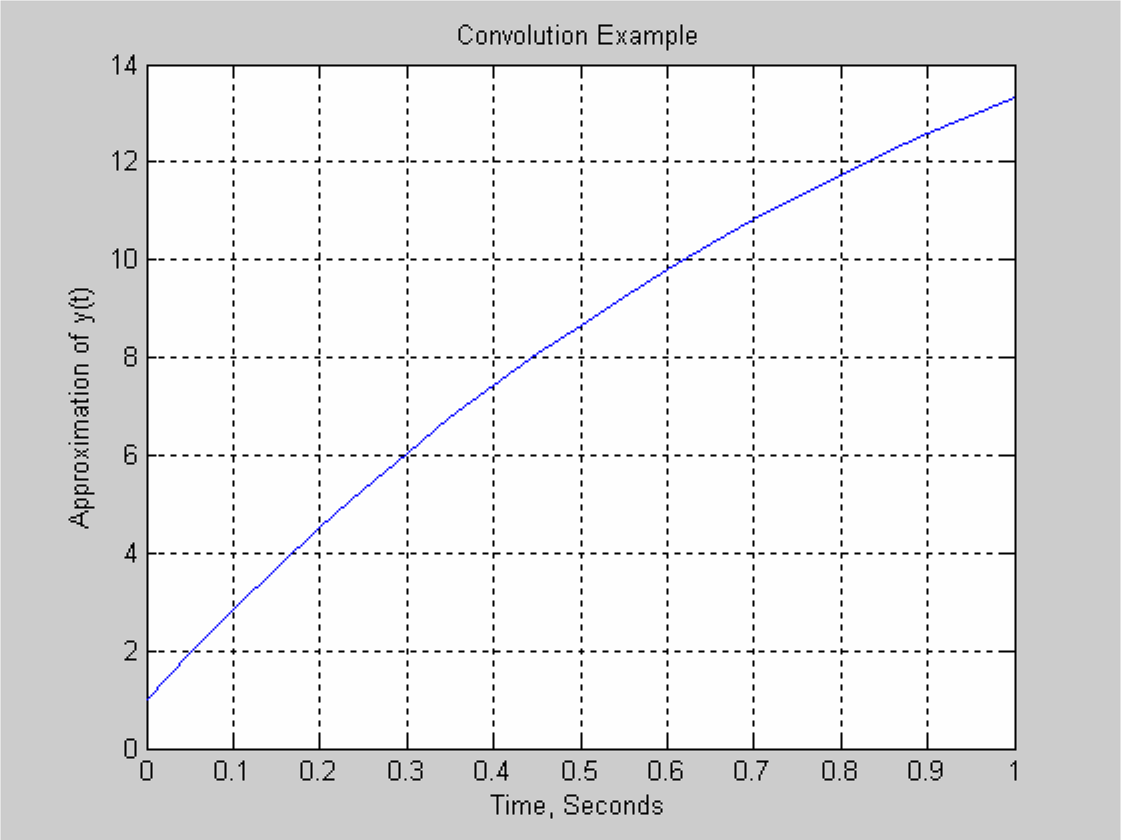
$$y(t) = \frac{AV}{a}.(1-e^{-at}).u(t) \qquad (3.14)$$

% Convolution of decaying exponential with a unit-step function.

```
t=0:.05:1
h=exp(-1*t)
```

```
x= ones(size(t))
y=conv(h,x)

plot(t,y(1:21))
title('Numerical convolution');
xlabel('Time, Seconds'); ylabel('Approximation of y(t)')
grid; axis
```



Example 3.5: Convolution of functions with a collection of impulses.
Let the input and the system functions be:

$$x(t) = G_{2a}((t-a)/2a) = rect((t-a)/2a) \quad \text{and} \quad h(t) = d(t+2a) - d(t-2a)$$

$$y(t) = rect((t-a)/2a) * d(t+2a) - rect((t-a)/2a) * d(t-2a)$$

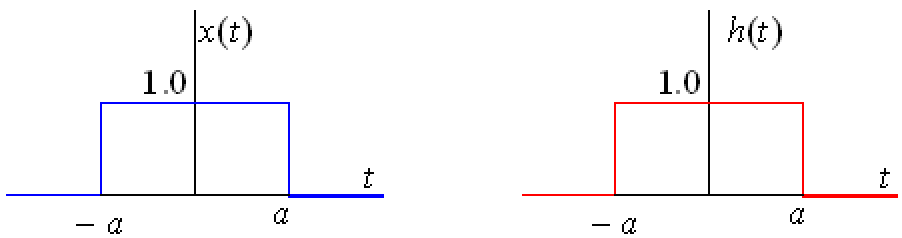
Using the properties of delta functions:

$$y(t) = rect[\frac{(t+2a)-a}{2a}] - rect[\frac{(t-2a)-a}{2a}] = rect[\frac{t+a}{2a}] - rect[\frac{t-3a}{2a}]$$

First rectangle of length $2a$ is centered at -a and the second one is an inverted rectangle of length $2a$ again but centered at $3a$.

Example 3.6: Convolution of two finite duration gate (rectangular) functions.
Task: Evaluate

$$y(t) = rect(t/2a) * rect(t/2a) \tag{3.15}$$



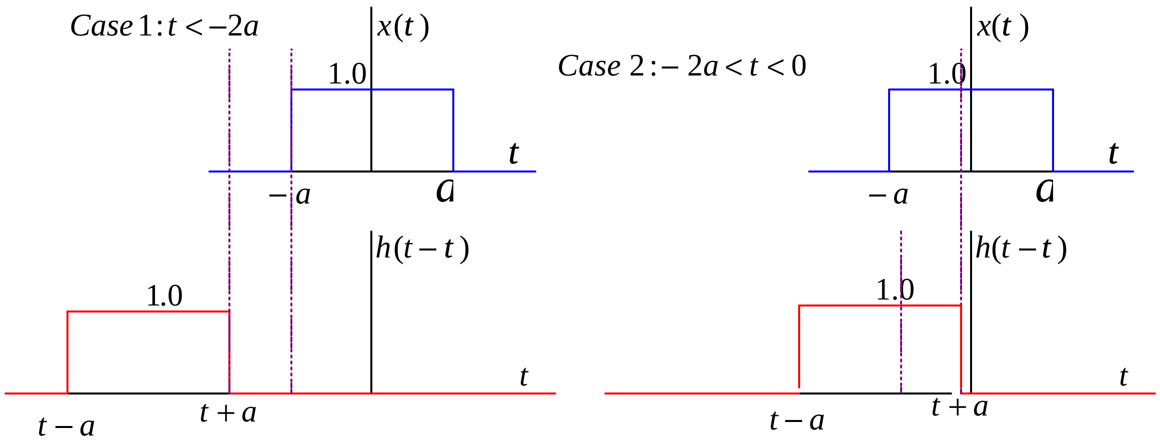
Both of these functions can be represented in terms of unit-step functions:

$$x(t) = u(t+a) - u(t-a) \tag{3.16}$$

When we substitute (3.16) into (3.15) we have:

$$y(t) = \int_{-\infty}^{\infty} [u(t+a) - u(t-a)].[u(t-(t+a)) - u(t-(t-a))]dt \tag{3.17}$$

As in Example 3.4, we need to determine segments of the above integral for which two brackets have non-zero product. Careful observation and with the following graphical support we see that there are four distinct cases.



Case 1: t<-2a:

There is no overlapping segments of the two pulses and the integral in (3.18) would yield 0.

$$y(t) = 0 \quad for\ t < -2a \tag{3.18a}$$

Case 2: -2a<t<0:

The interval between $-a$ and $t+a$ are common to both pulses then the integral becomes:

$$y(t) = \int_{-a}^{t+a} 1.dt = t + 2a \tag{3.18b}$$

By sliding the lower pulse (the system function) in above figure to the left we get two other cases.

Case 3: 0<t<2a:

The interval between t-a and a are common to both pulses and we get:

$$y(t) = \int_{t-a}^{a} 1.dt = 2a - t \tag{3.19}$$

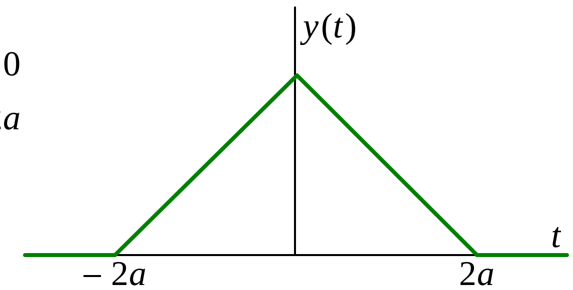
Case 4: t>2a:

Again there is no overlapping segments of the two pulses and the output would be zero.

$$y(t) = 0 \quad for\ t > 2a \tag{3.20}$$

All of these cases can be written in a compact form:

$$y(t)=\begin{cases} t+2a & if\ -2a<t<0 \\ -t+2a & if\ 0<t<2a \\ 0 & Otherwise \end{cases}$$

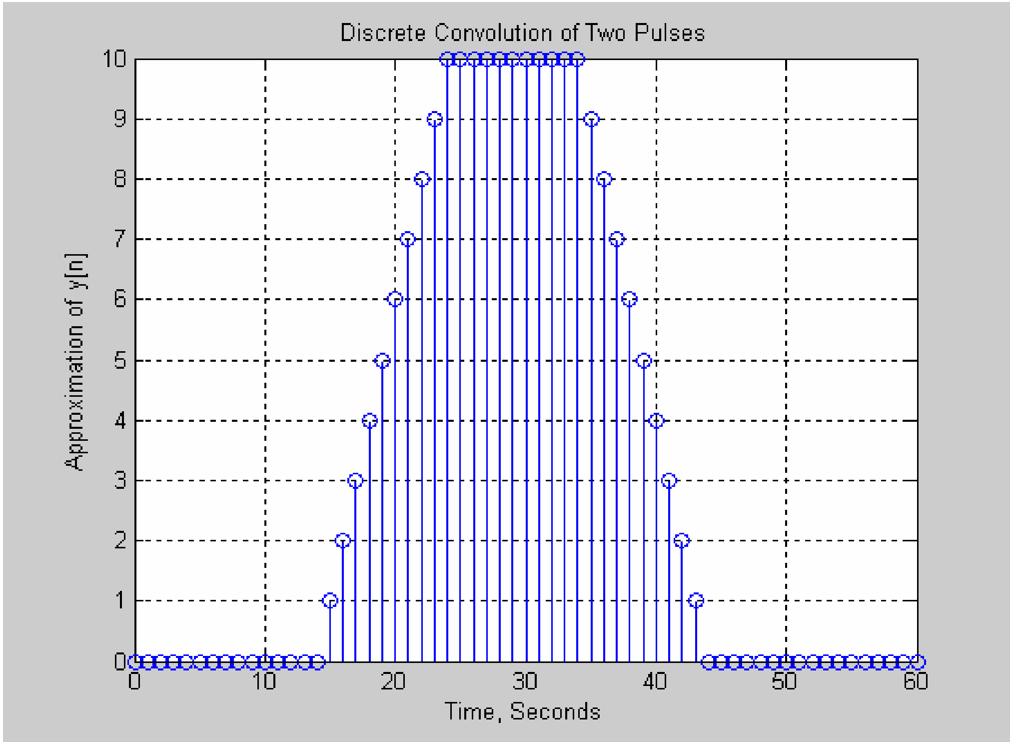


We can conclude that the convolution of two identical pulses is a triangle. What would be the shape of two different size rectangles?

% Convolution of decaying exponential with a unit-step function.

```
n=0:60;
x= zeros(size(n)); x(6:15)=1;
h=zeros(size(n)); h(11:30)=1;
y=conv(h,x)

stem(n,y(1:61));
title('Discrete Convolution of Two Pulses');
xlabel('Time, Seconds');
ylabel('Approximation of y[n]')
grid; axis
```



3.2.4 Linear Convolution Examples (Tabular Form):

Example 3.7: Consider the following system and signal sequences:

$$h[n]=\{-2,\overline{2},0,-1,1\} \qquad\qquad x[n]=\{-1,\overline{3},-1,-2\}$$

Note these two sequences have different lengths as in Example 3.6. It is not difficult to see that the output sequence $y[n]$ will be eight samples long in the interval $-2 \le n \le 5,$ zero elsewhere. Let us verify that with a linear convolution table.

n	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**	**5**
x[n+1]	-1	3	-1	-2				
x[n]		-1	3	-1	-2			
x[n-1]			-1	3	-1	-2		
x[n-2]				-1	3	-1	-2	
x[n-3]					-1	3	-1	-2
h[-1]x[n+1]	2	-6	2	4				
h[-0]x[n]		-2	6	-2	-4			
h[1]x[n-1]			0	0	0	0		
h[2]x[n-2]				1	-3	1	2	
h[3]x[n-3]					-1	3	-1	-2
Y[n]	2	-8	8	3	-8	4	1	-2

The last row or the output is: $y[n]=\{2,-8,\overline{8},3,-8,4,1,-2\}$ and $y[n]=0 \quad if \; n \le -3; \quad n \ge 6.$

3.4 Periodic (Circular) Convolution Process

In many applications, we are faced with the convolution of two periodic sequences, $x[n]\; and\; h[n],$ with or without a common period N. The method discussed below is geared to handle the common period case. If the periods are not common, then there are approaches to deal with the issue: (i) Find the smallest common product (SCD) of the two periods and perform the convolution over SCD. (2) Use Assume the longer of the two periods as the period and the other sequence is appended with zeros to bring the sequences to same length. This last approach is naturally violating the periodicity of the smaller one, but it does not pose a major problem in many engineering designs.

Let us now define the periodic (circular) convolution:

$$y[n] = x[k] \otimes h[k] = \sum_{k=0}^{N-1} x[k]h[n-k] \tag{3.21}$$

where $\otimes$ represents this periodic or circular convolution operation and the sum is over N terms. (3.21) is periodic using the property: $h[n + rN] = h[n]$

$$y[n+rN] = \sum_{k=0}^{N-1} x[k]h[n+rN-k] = \sum_{k=0}^{N-1} x[k]h[n-k] = y[n] \tag{3.22}$$

Since the sum is a finite sum, we can write out the full expression is a straightforward expansion:

$$y[n] = x[0]h[n) + x(1)h[n-1] + x(2)h[n-2] + \cdots + x(N-1)h[n-N+1] \tag{3.23}$$

and use the tabular form to compute the circular convolution of two periodic functions.

Example 3.8: Consider the following system and signal sequences:

$$x[n] = \{1,2,0,-1\} \qquad h[n] = \{1,3,-1,-2\}$$

Note these two sequences have a common period of 4 samples. It is not difficult to see that the output sequence $y[n]$ will be again 4 samples long in the interval $0 \le n \le 3$ and repeat itself. Let us verify that with a circular convolution table.

n	**0**	**1**	**2**	**3**
x[n]	1	2	0	-1
x[n-1]	-1	1	2	0
x[n-2]	0	-1	1	2
x[n-3]	2	0	-1	1
h[0]x[n]	1	2	0	-1
h[1]x[n-1]	-3	3	6	0
h[2]x[n-2]	0	1	-1	-2
h[3]x[n-3]	-4	0	2	-2
$y_c[n]$	-6	6	7	-5

The last row or the output is: $\mathrm{y}[\mathrm{n}] = \mathrm{y}_c[\mathrm{n}+4] = \{-6,6,7,-5\}$.

3.5 Differential Equation Model for LTI Systems (Continuous Case)

A general ordinary differential equation (ODE) model for linear time-invariant (LTI) systems is defined by:

$$\frac{d^N}{dt^N}y(t)=\sum_{i=0}^{M}b_i\frac{d^i}{dt^i}x(t)-\sum_{j=0}^{N-1}a_j\frac{d^j}{dt^j}y(t) \tag{3.24}$$

Here are the descriptions of various terms above:

$a_N \equiv 1$ and it is not normally shown in the ODE.

$a_j =$ Real coefficients associated with past outputs of the system (N feedback terms).

$b_i =$ Real coefficients associated with all inputs to the system (M+1 feed-forward terms).

$N =$ Highest-order derivative of the output.

$M =$ Highest-order derivative of the input.

$System\ Order \equiv \max\{N,M\}$.

In systems sciences, the above differential equation is generally written in one of the two operational forms, i.e., in terms of differential operator D or of integral operator D^{-1}.

$$D^N y(t)=\sum_{i=0}^{M}b_iD^ix(t)-\sum_{j=0}^{N-1}a_jD^jy(t) \tag{3.25}$$

For simplicity and nice symmetrical behavior let us assume that *N=M=System Order*.

$$D^N y(t)=\sum_{i=0}^{N}b_iD^ix(t)-\sum_{j=0}^{N-1}a_jD^jy(t) \tag{3.26}$$

In order to solve (3.26) we need N initial conditions for the output: $\{y(0),y'(0),\cdots,y^{N-1}(0)\}$. For EE applications, it is more common to replace the above ODE with an equivalent integral equation and use the integral operator D^{-1}.

$$D^{-N}\{D^N y(t)\}=D^{-N}\{\sum_{i=0}^{N}b_iD^ix(t)-\sum_{i=0}^{N-1}a_iD^iy(t)$$

$$y(t)=\{\sum_{i=0}^{N}b_iD^{i-N}\}x(t)-\{\sum_{i=0}^{N-1}a_iD^{i-N}\}y(t)$$

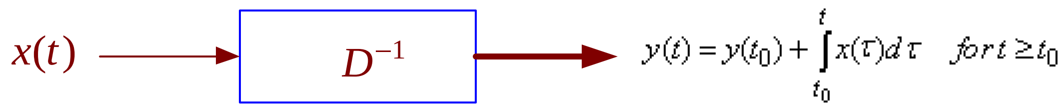
$$y(t) = b_N x(t) + \sum_{i=0}^{N-1} [b_i D^{i-N} x(t) - a_i D^{i-N} y(t)] \tag{3.27}$$

If we collect identical terms into common order pairs we obtain another frequently observed form:

$$y(t) = b_N x(t) + D^{-N}[b_0 x(t) - a_0 y(t)] + D^{-N+1}[b_1 x(t) - a_1 y(t)] + \cdots + D^{-1}[b_{N-1} x(t) - a_{N-1} y(t)] \tag{3.28}$$

We have canonical (standard) implementation forms based on simple building blocks for these last two equations (3.27) and (3.28) in terms of basic system building blocks:

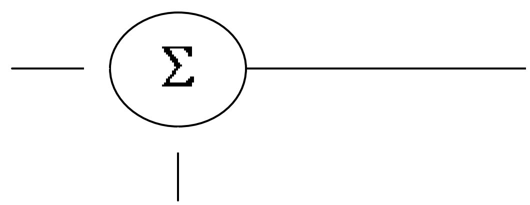
1. Integrator D^{-1}:



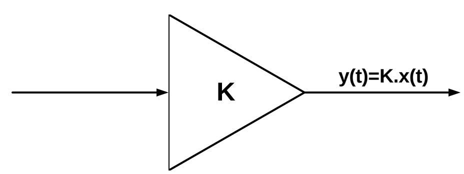
If $y(t_0) = 0$ then the system is said to be ***at rest*** and we have the usual case:

$$y(t) = \int_{t_0}^{t} x(t) dt \quad for\ t_0 \geq 0 \tag{3.29}$$

2. Adder (Accumulator) $\sum$:



3. Scalar Multiplier K:



We will next use these three fundamental building blocks to implement systems expressed in terms of ODE and/or integral equations.

Example 3.9: Implement and solve the following system using building blocks.

$$\frac{d}{dt}y(t)+a.y(t)=b.x(t) \tag{3.30}$$

Since the highest-order derivative is dy/dx then the order of this system is "1." Let us convert this first-order ODE into operational form:

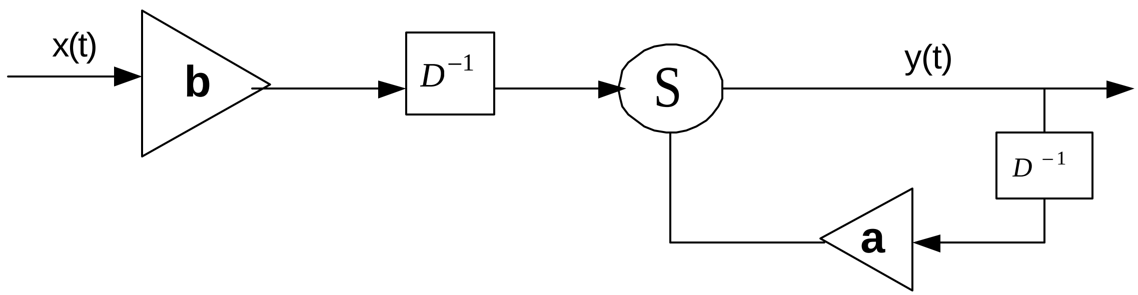
$$Dy(t)+a.y(t)=b.x(t) \tag{3.31}$$

$$D^{-1}[Dy(t)+a.y(t)]=D^{-1}bx(t)$$

$$y(t)+a.D^{-1}y(t)=b.D^{-1}x(t)$$

Finally, we have the form ready for implementation using the set of blocks discussed above:

$$y(t)=-a.D^{-1}y(t)+b.D^{-1}x(t) \tag{3.32}$$



ODE has a total solution composed of a homogeneous solution (natural response) $y_h(t)$ and a particular solution (forced response) $y_p(t)$:

$$y(t)=y_h(t)+y_p(t) \tag{3.33}$$

Homogeneous Equation and its solution:

$$Dy_h(t)+ay_h(t)=0 \tag{3.34}$$

Assume a solution of the form: $y_h(t) = C.e^{-at}$ and a solution of the form for the particular part:

$$y_p(t) = \int_{t_0}^{t} e^{-a(t-t)}.b.x(t)dt \quad for\ t \geq t_0 \tag{3.35}$$

Now let us substitute these two solutions into (3.33):

$$y(t) = C.e^{-at} + b.\int_{t_0}^{t} e^{-a(t-t)}.x(t)dt \tag{3.36}$$

C can be obtained by evaluating this at IC:

$$Y_0 = C.e^{-at} + b.\int_{t_0}^{t_0} e^{-a(t-t)}.x(t)dt = C.e^{-at} + 0 \quad \Rightarrow C = Y_0.e^{at_0}$$

and the total solution:

$$y(t) = Y_0.e^{at_0}.e^{-at} + [b.\int_{t_0}^{t} e^{-a(t-t)}.x(t)dt].u(t-t_0) \tag{3.37}$$

3.6 Canonical Forms for LTI Continuous Systems

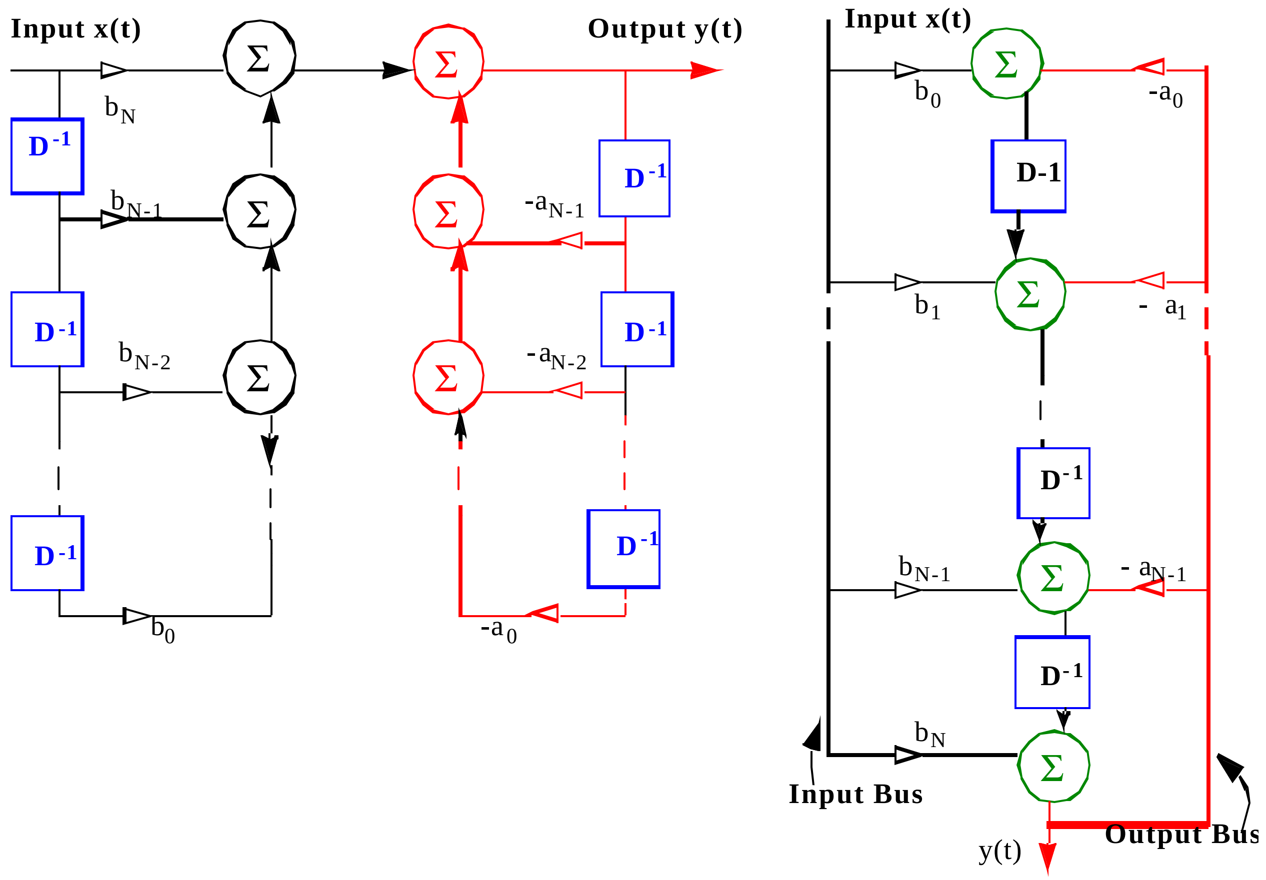
Consider any of the known general ODE representation of an LTI system as formulated in (3.24), (3.26), (3.27), or (3.28). To have a form to reference let us rewrite (3.28)

$$y(t) = b_N x(t) + D^{-N}[b_0 x(t) - a_0 y(t)] + D^{-N+1}[b_1 x(t) - a_1 y(t)] + \cdots + D^{-1}[b_{N-1} x(t) - a_{N-1} y(t) \tag{3.28}$$

and implement it using two different canonical (standard) structures, Canonical Direct Form I and I/O-Bus architecture. It is not difficult to see that there are two paths: *feed-forward branches and feedback loops*.

Canonical Direct Form I

I/O-Bus Form



Example 3.10: Assume that the following system is at rest at $t=0$; i.e., all initial values are zero for $t=0$.

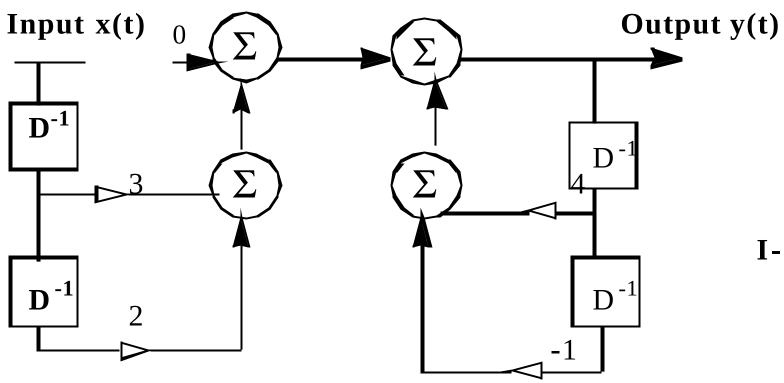
$$\frac{d^2}{dt^2}y(t)-4\frac{d}{dt}y(t)+y(t)=3\frac{d}{dt}x(t)+2x(t)$$

$$D^2y(t)=3Dx(t)+2x(t)+4Dy(t)-y(t)$$

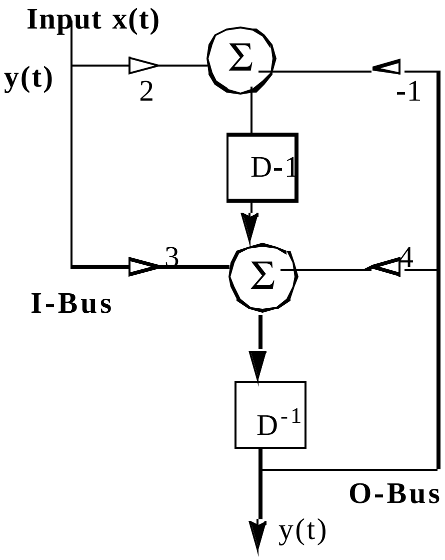
$$D^{-2}[D^2y(t)]=D^{-2}[3Dx(t)+2x(t)+4Dy(t)-y(t)]$$

$$y(t)=D^{-1}[3x(t)+4y(t)]+D^{-2}[2x(t)-y(t)]$$

Canonical Form I:



I/O Bus Model:



3.7 Difference Equation Model for LTI Systems (Discrete Case)

As in the case of continuous signals and systems, discrete linear time-invariant systems can be expressed in terms of DE of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \textit{for } n \geq 0 \tag{3.38}$$

Similar to the previous case, the descriptions of various terms above:

a_j = Real coefficients associated with past outputs of the system (*N feedback terms*).

b_i = Real coefficients associated with all inputs to the system (*M+1 feed-forward terms*).

N = Highest-order difference in the output sequence.

M = Highest-order difference in the input sequence.

$\textit{System Order} \equiv \max\{N, M\}$. If the system is causal we must have: $M \leq N$.

Let us define a difference operator:

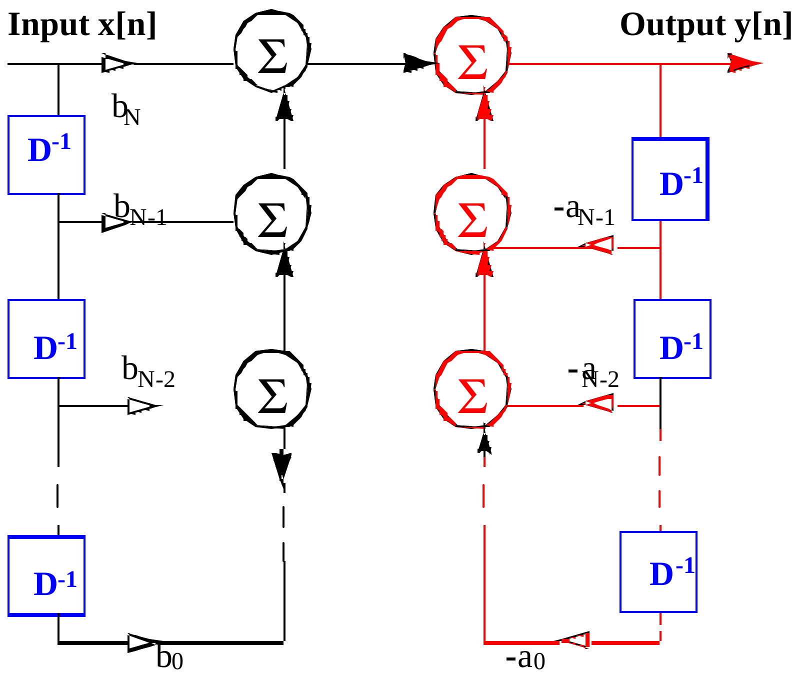
$$D^k y[n] = y[n-k] \tag{3.39}$$

$$\sum_{k=0}^{N} a_k D^k y[n] = \sum_{k=0}^{M} b_k D^k x[n] \quad \textit{for } n \geq 0 \tag{3.40}$$

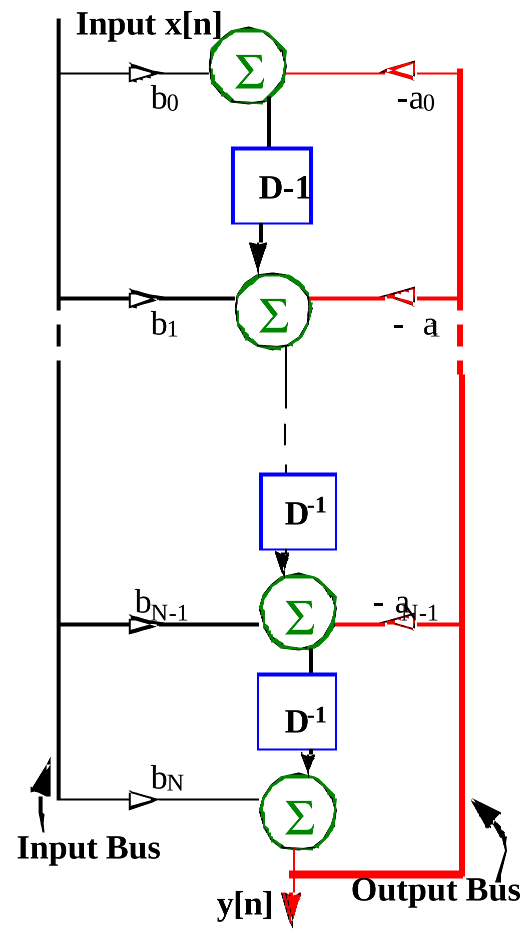
However, the output is implicitly expressed in (3.40); buried among the feedback terms. It is usually expressed in the following form:

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k D^k x[n] - \sum_{k=1}^{N} a_k D^k y[n]\right) \qquad \textit{for } n \geq 0 \tag{3.41}$$

Canonical Direct Form I



I/O-Bus Form



It is clear from (3.41) and the canonical implementations, $x[n-k]$ are known at any given time. If we have done our job correctly then $y[n-k]$ are also known. Then $y[0]$ can be computed from:

$$y[0]=\frac{1}{a_0}(\sum_{k=0}^{M}b_k x[-k]-\sum_{k=1}^{N}a_k y[-k]) \qquad (3.42)$$

where $y[-k]$ are the initial conditions (IC). Next, we compute:

$$y[1]=\frac{1}{a_0}(\sum_{k=0}^{M}b_k x[1-k]-\sum_{k=1}^{N}a_k y[2-k]) \qquad (3.43)$$

Similarly, we can compute all future outputs. Note that we need to do that an iterative (recursive) fashion; i.e., it is not possible to

Example 3.11: Given $y[-1]=1 \quad and \quad y[-2]=0$ compute **RECURSIVELY** a few terms of the following 2nd order DE:

$$y[n]=\frac{3}{4}y[n-1]-\frac{1}{8}y[n-2]+(\frac{1}{2})^n$$

$$y[0]=\frac{3}{4}y[-1]-\frac{1}{8}y[-2]+(\frac{1}{2})^0=\frac{3}{4}+0+1=\frac{7}{4}$$

$$y[1]=\frac{3}{4}y[0]-\frac{1}{8}y[-1]+(\frac{1}{2})^1=\frac{27}{16}$$

$$y[2]=\frac{3}{4}y[1]-\frac{1}{8}y[0]+(\frac{1}{2})^2=\frac{83}{64}$$

$$\vdots$$

3.7 Homogeneous and Particular Solutions for Discrete LTI Systems

Generic DE problems cannot be solved recursively unless they happen to have a compact closed form. As in the case of ODE, we attempt to solve them by finding (a) a homogeneous solution and (b) the particular solution.

3.7.1 Homogeneous Solution: It is given by solving:

$$\sum_{k=0}^{N}a_k y[n-k]=0 \qquad (3.44)$$

$$y_h[n]=A.a^n$$

Substitute into the DE (3.44)

$$\sum_{k=0}^{N} a_k A a^{n-k} = 0$$ Since $A \neq 0$, the solution must satisfy:

$$\sum_{k=0}^{N} a_k a^{n-k} = 0 \tag{3.45}$$

Values satisfying (3.45) are the characteristic values (eigenvalues) and there are N of them, which may or may not be distinct. If they are distinct, the corresponding characteristic solutions are independent and they are obtained as a linear combination of the terms like:

$$y_h[n] = A_1 a_1^n + A_2 a_2^n + \cdots + A_N a_N^n \tag{3.46}$$

If any of the roots are repeated, then we can generate N independent solutions by multiplying corresponding characteristic solution by the appropriate power of n. For instance, if a_1 has a multiplicity of P_1, then we assume a solution of the form:

$$y_h[n] = A_1 a_1^n + A_2 n a_1^n + \cdots + A_{P_1} n^{P_1 - 1} a_1^n + A_{P_1+1} a_{P_1+1}^n + \cdots + A_N a_N^n \tag{3.47}$$

3.7.2 Particular Solution: Assume that $\bar{y}[n]$ is a particular solution to a special case:

$$\sum_{k=0}^{N} a_k \bar{y}[n-k] = x[n] \tag{3.48}$$

then the overall particular solution is found by a superposition:

$$y_P[n] = \sum_{k=0}^{M} b_k \bar{y}[n-k] \tag{3.49}$$

To find $\bar{y}[n]$, we assume it is a linear combination of $x[n]$ and its delayed versions.

- If $x[n]$ is a constant then $x[n-k]$ is also constant. Thus, $\bar{y}[n]$ is another constant.
- If $x[n]$ is an exponential function if the form: b^n, then $\bar{y}[n]$ is similarly an exponential.
- If $x[n]$ is a sinusoid:

  $x[n] = \sin \Omega_0 n$ then $x[n-k] = \sin \Omega_0 (n-k) = \cos \Omega_0 k . \sin \Omega_0 n - \cos \Omega_0 n . \sin \Omega_0 k$

  $\bar{y}[n] = A . \sin \Omega_0 n + B . \cos \Omega_0 n$

Example 3.12: Given $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2.\sin\frac{np}{2}$ with IC: $y[-1]=2 \quad and \quad y[-2]=4$

Part A: Particular solution: Assume a solution:

$$y[n]=A.\sin\frac{np}{2}+B.\cos\frac{np}{2}$$

$$y_P[n-1]=A.\sin\frac{(n-1)p}{2}+B.\cos\frac{(n-1)p}{2}=-A\cos\frac{np}{2}+B\sin\frac{np}{2}$$

$$y_P[n-2]=-A.\cos\frac{(n-1)p}{2}+B.\sin\frac{(n-1)p}{2}=-A\sin\frac{np}{2}-B\cos\frac{np}{2}$$

Let us substitute these into the DE, which must be satisfied in order for this to be solution:

$$(A-\frac{3}{4}B-\frac{1}{8}A).\sin\frac{np}{2}+(B+\frac{3}{4}A-\frac{1}{8}B).\cos\frac{np}{2}=2.\sin\frac{np}{2}$$

Let us equate terms of the same form:

$$A-\frac{3}{4}B-\frac{1}{8}A=2$$

$$B+\frac{3}{4}A-\frac{1}{8}=0 \quad \Rightarrow \quad A=\frac{112}{85} \quad and \quad B=-\frac{96}{85}$$

$$y_P[n]=\frac{112}{85}\sin\frac{np}{2}-\frac{96}{85}\cos\frac{np}{2}$$

Part B: Homogeneous solution: Write the characteristic equation:

$$1-\frac{3}{4}a^{-1}+\frac{1}{8}a^{-2}=0 \quad \Rightarrow \quad a_1=\frac{1}{4} \quad and \quad a_2=\frac{1}{2}$$

resulting an a homogeneous solution:

$$y_h[n]=A_1(\frac{1}{4})^n+A_2(\frac{1}{2})^n$$

Part C: Total solution:

$$y[n]=A_1(\frac{1}{4})^n+A_2(\frac{1}{2})^n+\frac{112}{85}\sin\frac{np}{2}-\frac{96}{85}\cos\frac{np}{2}$$

If we substitute the given ICs to this last expression we could obtain that:

$$A_1 = -\frac{8}{17} \qquad A_2 = \frac{13}{5}$$

and

$$y[n] = -\frac{8}{17}(\frac{1}{4})^n + \frac{13}{5}(\frac{1}{2})^n + \frac{112}{85}\sin\frac{np}{2} - \frac{96}{85}\cos\frac{np}{2}$$

3.8 Impulse Response Computation of Discrete LTI Systems

If we derive the difference equation of (3.38) with a train of impulses, we have

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k d[n-k] \qquad \text{with: } y[-1] = y[-2]\cdots = 0 \tag{3.50}$$

1. For $n > M$, the right hand side is zero, thus we get a homogeneous equation.
2. N initial conditions (IC) to solve this equation are: $\{y[M), y[M-1], \cdots, y[M-N+1]\}$.
3. To be meaningful this system must be causal: $N \geq M$ and we have to compute only the terms: $y[0], y[1], \cdots, y[M]$.
4. By successively letting n be $0,1,2,\cdots,M$ in (3.50) we obtain a set of $M+1$ equations:

$$\sum_{k=0}^{j} a_k y[n-k] = b_j \qquad for \quad j = 0,1,2,\cdots,M \tag{3.51}$$

This is normally written in a matrix form:

$$\begin{bmatrix} a_0 & 0 & . & . & 0 \\ a_1 & a_0 & 0 & . & 0 \\ a_2 & a_1 & a_0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ a_M & a_{M-1} & . & . & a_0 \end{bmatrix} . \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ . \\ . \\ y[M] \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ . \\ . \\ b_M \end{bmatrix} \tag{3.52}$$

After solving this system for initial conditions $y[0], y[1], \cdots, y[M]$, we obtain the impulse response of the system as the solution of the homogeneous equation:

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \quad for \;\; n > M \tag{3.53}$$

Example 3.13: Consider the system described by the difference equation:

$$y[n] = x[n] + \frac{1}{3}x[n-1] + \frac{5}{4}y[n-1] - \frac{1}{2}y[n-2] + \frac{1}{16}y[n-3]$$

Here $N = 3,\; M = 1$. Order 3 homogeneous equation:

$$y[n] - \frac{5}{4}y[n-1] + \frac{1}{2}y[n-2] - \frac{1}{16}y[n-3] = 0 \qquad n \geq 2$$

The characteristic equation:

$$1 - \frac{5}{4}a^{-1} + \frac{1}{2}a^{-2} - \frac{1}{16}a^{-3} = 0$$

The roots of this third order polynomial is: $a_1 = a_2 = 1/2 \quad a_3 = 1/4$ and

$$y_h[n] = h[n] = A_1(\frac{1}{2})^n + A_2 n(\frac{1}{2})^n + A_3(\frac{1}{4})^n, \quad n \geq 2$$

Let us assume $y[-1] = 0$ then (3.52) for this case becomes:

$$\begin{bmatrix} a_0 & 0 \\ a_1 & a_0 \end{bmatrix} . \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} \;\; \Rightarrow \;\; \begin{bmatrix} 1 & 0 \\ -5/4 & 1 \end{bmatrix} . \begin{bmatrix} y[0] \\ y[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 1/3 \end{bmatrix} \;\; \Rightarrow y[0] = 1;\; y[1] = 19/12$$

with these we have the impulse response of this system:

$$h[n] = -\frac{4}{3}(\frac{1}{2})^n + \frac{10}{3}n(\frac{1}{2})^n + \frac{7}{3}(\frac{1}{4})^n, \quad n \geq 0$$