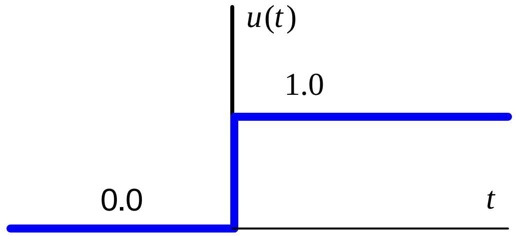
Chapter 2: CONTINUOUS AND DISCRETE SIGNALS AND SYSTEMS
2.1 CONTINUOUS SIGNALS

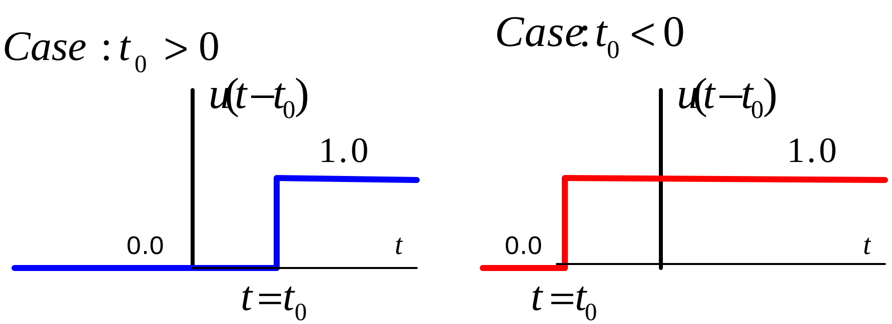
2.1.1 Unit Step Function: *u(t)*

$$u(t) = \begin{cases} 1 & if \quad t > 0 \\ 0 & Otherwise \end{cases} \tag{2.1}$$



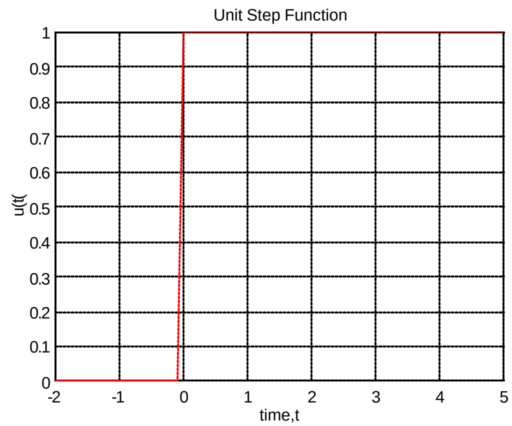
Example 2.1: Generic unit step function (electronic switch turned on at t_0):

$$u(t - t_0) = \begin{cases} 1 & if \quad t > t_0 \\ 0 & Otherwise \end{cases} \tag{2.2}$$



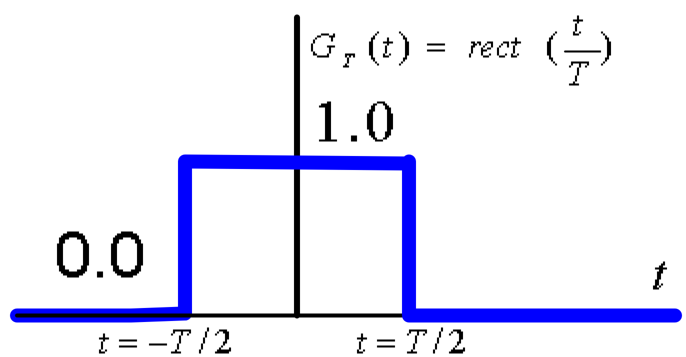
% Matlab Demonstration for Example 2.1: Step Function

```
% Make t a vector of 71 points
t=-2:0.1:5;
% Set f as a vector of zeros
f=zeros(size(t));
% Set final 50 points of f to 1
f(21:71)=ones(size(t(21:71)));
%plot
plot(t,f,'r')
title('Unit Step Function'); xlabel('time,t');
ylabel(' u(t(')); grid;
```



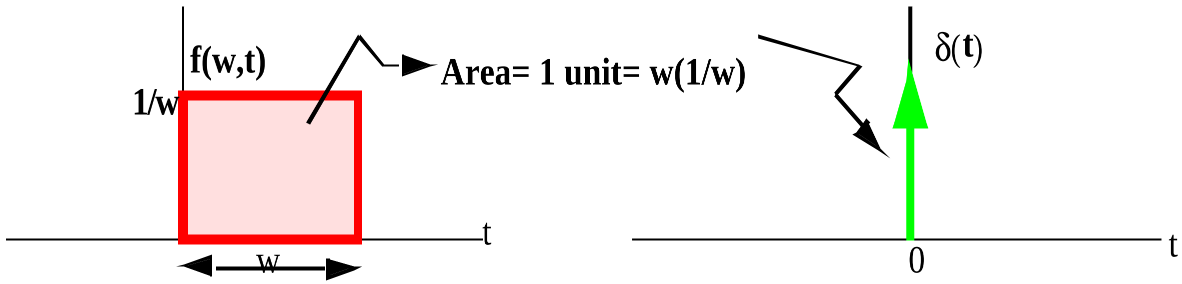
2.1.2 Pulse Function (Gate Function, Time Window):

$$rect(\frac{t}{T}) = G_T(t) = \begin{cases} 1 & if \quad t < |T/2| \\ 0 & Otherwise \end{cases} \qquad (2.3)$$



2.1.3 Unit Impulse (Delta) function: It is a limit of a pulse with an area of One Unit:

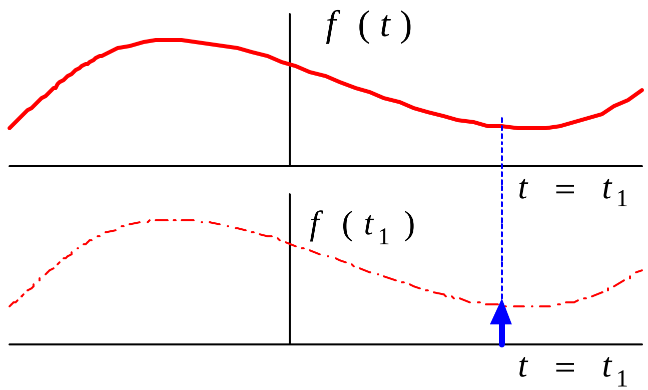
$$d(t) = \lim_{w \to 0} f(w,t) \qquad (2.4)$$



This description of we have presented above is a conceptual definition. However, the formal definition of a unit impulse function is normally done through what is known as the *Sifting Theorem* in the field of applied mathematics.

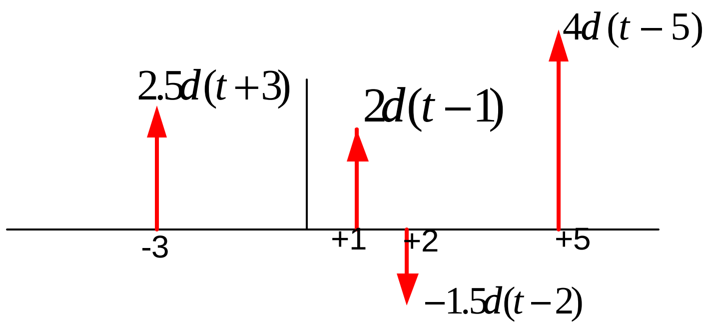
Sifting Theorem definition of a unit impulse (delta) function:

$$f(t_1) = \int_{-\infty}^{\infty} f(t).d(t - t_1)dt \qquad (2.5)$$



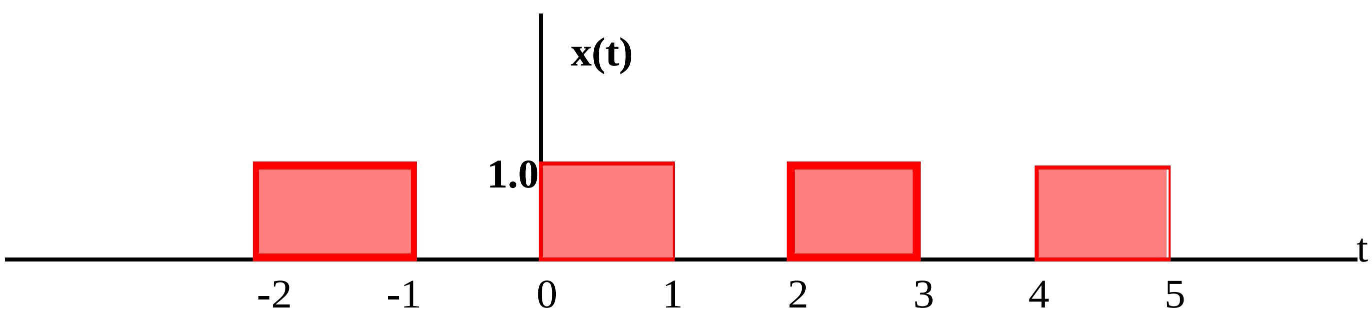
Example 2.2: Here we will compose a rather general signal from integration (summation) of number of impulses. The principle is to integrate all the applicable impulses:

$$f(t) = \int_{-\infty}^{\infty} f(t).d(t-t)dt \qquad (2.6)$$



2.1.4 Periodic Signals: A continuous-time signal $x(t)$ is periodic with a period T if there is a positive nonzero number, such that $x(t) = x(t+kT)$, where k = *...-3,-2,-1,0,1,2,3,....* is an integer.

Example 2.3: Periodic repetition of unit-pulses with a period of T =*2.0* seconds



2.1.5 Signals from sinusoid family:

$$x(t) = A.Cos(\frac{2p}{T}t + f) = ACos(w_0 t + f) = ACos(2pf_0 t + f) \tag{2.7}$$

where

- A= Amplitude
- f = Phase at *t=0*
- f_0 =Fundamental frequency in Hz.
- $w = \frac{2p}{T} = 2p.f_0$ Angular frequency in radians/s.

Example 2.4: Harmonically Related Sinusoids:

$$j_k(t) = e^{j2pkf_0 t} \quad for \quad k = \cdots,-2,-1,0,1,2,\cdots \tag{2.8}$$

with

$$Cos(2pkf_0 t) = \mathrm{Re}\{j_k(t)\} \quad \text{and} \quad Sin(2pkf_0 t) = \mathrm{Im}\{j_k(t)\} \tag{2.9}$$

It is worth noting that here we have used the trigonometric identity: $e^{jq} = Cosq + jSinq$ and it is important to note that every harmonic $j_k(t)$ have the same period: $T = 2p / w_0 = 1/ f_0$ due to the fact that functions periodic in *T* are also periodic with any positive multiple of T.

2.1.6 Composite Signals: In many applications the signal cannot be represented by a simple mathematical function. Instead, it is expressed as a composition of a number of known functions. One example for such signals is a sinusoid which is turned "ON" at a given time and turned "OFF" later as in the case of an electronic switch.

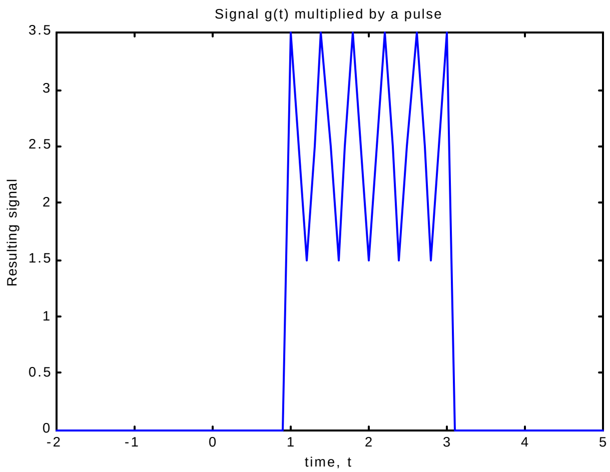
Example 2.5: Let us construct one such signal.

$$g(t) = Cos(2pf_0 t) \qquad \text{and} \qquad p(t) = u(t-1) - u(t-3)$$

Task is to form the following composite signal using matlab: $x(t) = p(t).[2.5 - g(t)]$

% Matlab Demonstration: A sinusoid multiplied by a pulse

```
t=-2:0.1:5;
f=zeros(size(t));
f(31:51)=2.5-cos(5*pi*t(31:51));
plot(t,f,'b');
title('Sinosoid multiplied by a pulse');
xlabel('time, t'); ylabel('Resulting signal');
axis;
```



2.1.7 Exponentially Modulated Signals: For these signals the generic equation is given by:

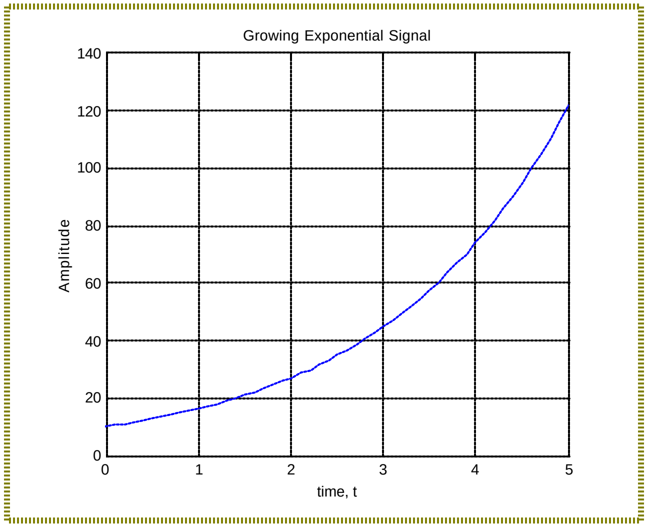
$$x(t) = A.e^{at}.Cos(2pf_0 t + q) \qquad (2.10)$$

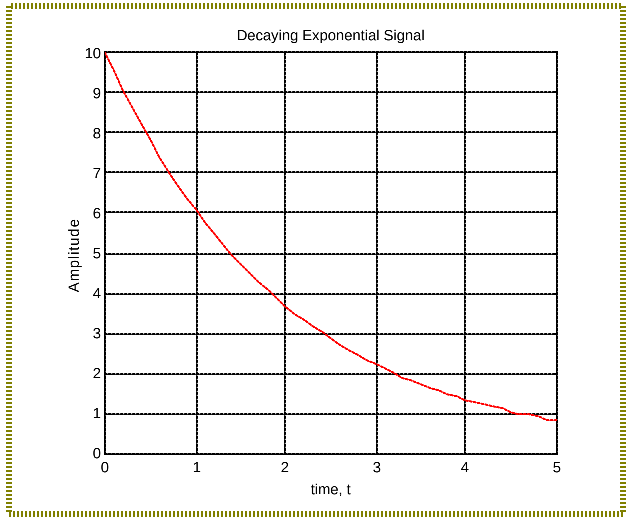
- A = Amplitude of the signal
- $a = \begin{cases} Attenuation \quad if \ it \ is \ less \ than \ zero \quad and \ t > 0 \\ Amplification \quad if \ it \ is \ greater \ than \ zero \quad and \ t > 0 \end{cases}$
- $Cos(2pf_0 t + q) = Sinusoidal \ Oscillation$ and $q = Phase angle \quad at \ t = 0$.

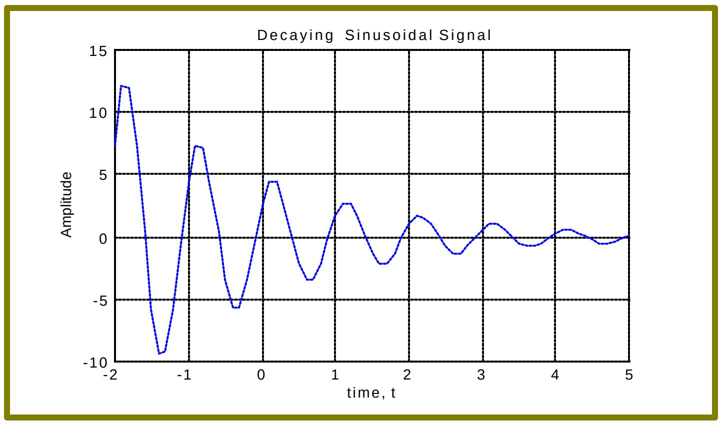
% Example 2.6: Growing and Decaying Exponential Signals:

```
t=0:0.1:5;
f=zeros(size(t));
f= 10.*exp(0.5*t);
plot(t,f,'b'); grid;
title('Growing Exponential Signal');
xlabel('time, t'); ylabel('Amplitude')
```

```
t=0:0.1:5;
f=zeros(size(t));
f= 10.*exp(-0.5*t);
plot(t,f,'r'); grid;
title('Decaying Exponential Signal');
xlabel('time, t'); ylabel('Amplitude')
```







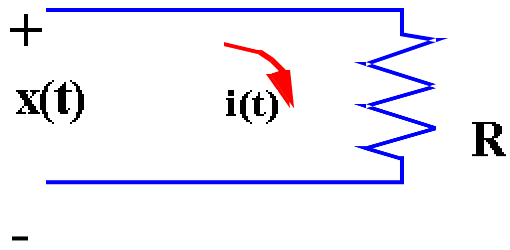
% Example 2.7: Exponentially Modulated Decaying Signal

```
ts=-2; tf=5;  %Start and Stop times
dt = 0.1;     % Time increment
t=ts:dt:tf;
f1=5*exp(-0.5*t) ;
f2= cos(2*pi*t-1) ;
f=f1.*f2;
```

```
%Plotting
plot(t,f,'b'); grid
title('Decaying  Sinusoidal Signal');
 xlabel('time, t'); ylabel('Amplitude')
```

2.2 Some Definitions and Terminology for Continuous Signals

2.2.1 Energy and Power Signals: Consider the following resistive circuit:



$$i(t)=x(t)/R \quad Amps$$

The instantaneous power dissipated across this resistor is defined by:

$$P_{ins} = i^2(t).R = \frac{x^2(t)}{R} \quad Watts \tag{2.11a}$$

Similarly, the energy in the signal over a finite time interval $-L < t < L$ is defined by:

$$E_{2L} = \int_{-L}^{L} |x(t)|^2 dt \tag{2.11b}$$

and the total energy is found by forcing L to infinity:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad Joules \tag{2.11c}$$

On the other hand, the average power is the rate of change of energy over a time interval:

$$P = P_{av} = Lim_{L\to\infty}[\frac{1}{2L}.E_{2L}] = Lim_{L\to\infty}[\frac{1}{2L}.\int_{-L}^{L}|x(t)|^2 dt] \quad Watts \tag{2.12}$$

Important Observations:

1) Energy Signals: If $0 < E < \infty$; i.e., signal has a finite energy then *x(t)* is called an energy signal. Note: All energy signals have ZERO power.

2) Power Signals: If $0 < P < \infty$; i.e., signal has a finite power then *x(t)* is a power signal. Note: All power signals have INFINITE energy.

Example 2.8: Given $x(t) = A.e^{-t}$ Volts. Find energy and power content of this signal.

$$E = \int_{-\infty}^{\infty} \left|A.e^{-t}\right|^2 .dt = 2\int_0^{\infty} A.e^{-2t} .dt = \frac{2A^2}{-2} e^{-2t}\Big|_0^{\infty} = A^2 \qquad (2.13)$$

which has a finite value; then *x(t)* is an energy signal and it must have $P = P_{av} = 0$. Let us verify that:

$$P = Lim_{L\to\infty}[\frac{1}{2L}\int_{-L}^{L}\left|x(t)\right|^2 dt = Lim_{L\to\infty}[\frac{2A^2}{2L}\int_0^L e^{-2t} dt] = Lim_{L\to\infty}[\frac{A^2}{-2L} e^{-2t}\Big|_0^L]$$
$$= Lim_{L\to\infty}\frac{2A^2}{-2L}(e^{-2L} - e^{-0}) = 0 \qquad (2.14)$$

Example 2.9: Consider a sinusoidal signal: $x(t) = A.Sin(w_0 t + f)$ *with a period* $: T = \frac{2p}{w_0}$ find the power and energy content of this signal.

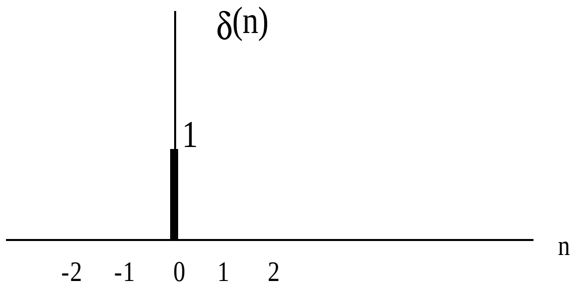
$$P_I = \int_0^T \frac{A^2}{T} Sin^2(w_0 t + f)dt = \frac{A^2}{T}\int_0^T [\frac{1}{2} - \frac{1}{2}Cos(2w_0 t + 2f)]dt = \frac{A^2}{T}.\frac{T}{2} = \frac{A^2}{2} \qquad (2.15)$$

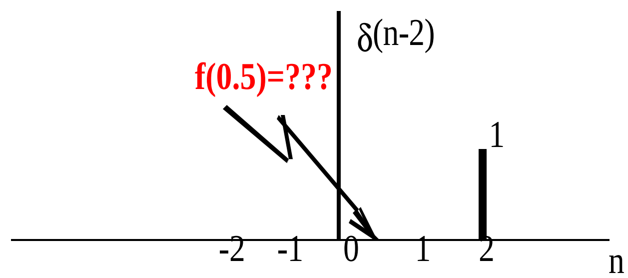
This is due to the fact that the second integral in the above equation goes to zero. Thus, we have shown that the power has a finite value as long as the period is finite. It is very easy to see that the energy is the sum of the energy contribution from each period:

$$E_{Total} = \frac{A^2}{2}T + \frac{A^2}{2}T + \cdots + \frac{A^2}{2}T \to \infty \qquad (2.16)$$

2.3 Discrete-Time Signals (Sequences)

In this case, the time stamp of signals are measured in discrete time instances. However, the amplitude could assume any value as before. Let us now consider few important signals of this type.





2.3.1 Unit-sample sequence:

$$d[n]=\begin{cases}1 & if\ n=0\\ 0 & Otherwise\end{cases} \quad \text{and} \quad d[n-k]=\begin{cases}1 & if\ n=k\\ 0 & if\ n\neq k\end{cases} \qquad (2.17)$$

Any discrete signal $f[n]$ can be represented by a sum of appropriately weighted unit-sample functions:

$$f[n]=\sum_{m=-\infty}^{\infty} f[m].d[n-m] \qquad (2.18)$$

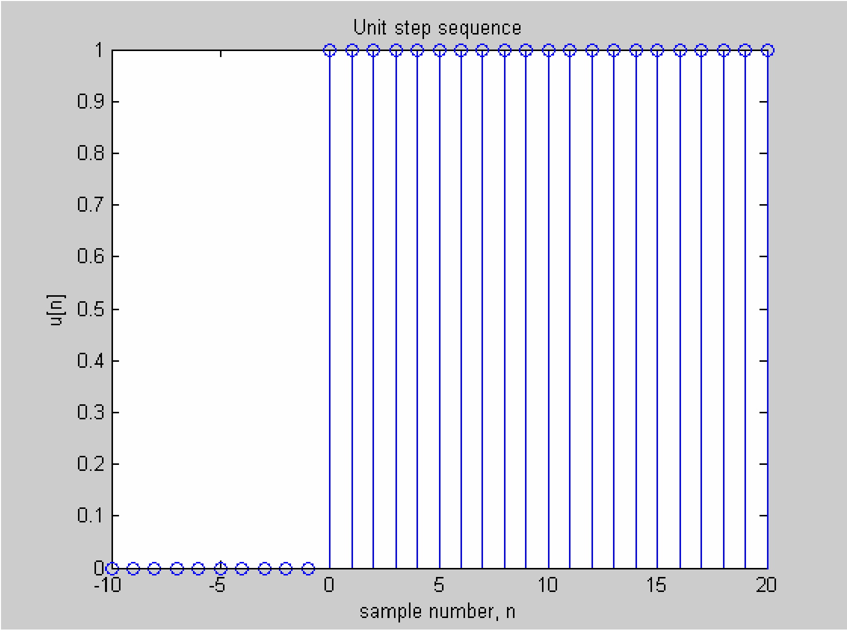
where $f[m]$ is the weight at location $n=m$ and the delta function represents the unit-sample to isolate the signal at that point & only at that point.

2.3.2 Unit-step and generic step sequences:

$$u[n]=\begin{cases}1 & if\ n\geq 0\\ 0 & Otherwise\end{cases}=\sum_{m=0}^{\infty} d[n-m] \quad \text{and} \quad u[n-n_0]=\begin{cases}1 & if\ n\geq n_0\\ 0 & if\ n<n_0\end{cases} \qquad (2.19)$$

%Example 2.10: Unit step sequence

```
n=-10:1:20;
f=zeros(n);
f(11:31)=1;
axis([-10,20,-1,2]);
stem(n,f,)
xlabel ('sample number, n');
ylabel ('u[n]’);
ptitle('Unit step sequence');
```



2.3.3 Ramp sequence:

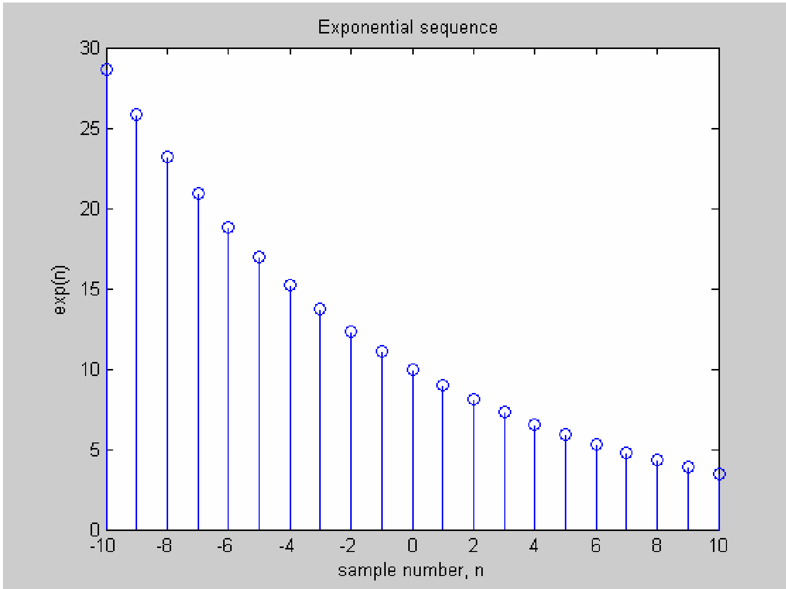
$$r[n] = B[n-n_0].u[n-n_0] \quad where \begin{cases} B = Slope\,of\ \ the\,ramp. \\ n_0 = Delay\,or\ shift\ in\ samples. \end{cases} \tag{2.20}$$

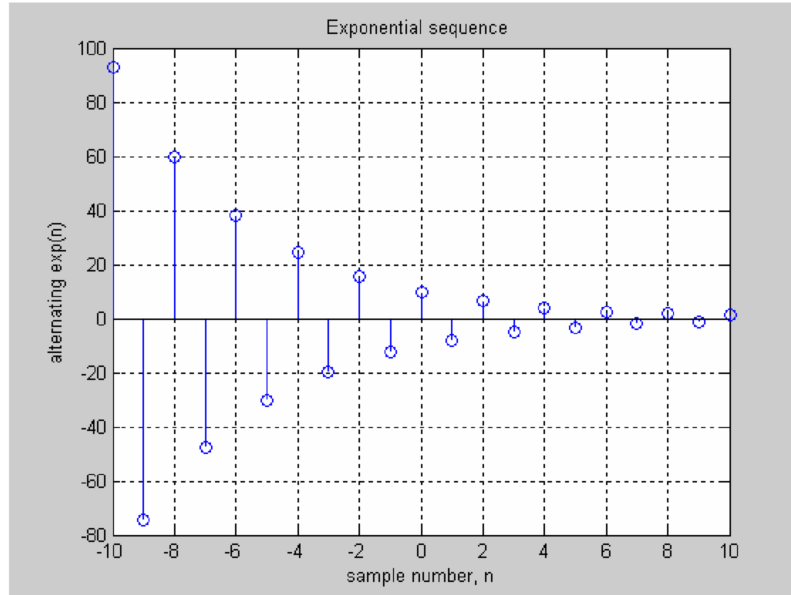
2.3.4 Exponential sequences (Real):

$$x[n] = A.a^n \quad where\ \ A = Amplitude\ \ and\ \ a = Base \tag{2.21}$$

%Example 2.11: Decaying Exponential sequence

```
n=-10:1:10;
exp =10*(.9).^n;
axis([-10 10 0 30]);
stem(n,exp)
xlabel('sample number, n'); ylabel ('exp(n)'); title('Exponential sequence ')
```





%Example 2.12: Alternating Exponential sequence

$$x[n] = A.a^n \quad where \quad A = 10; \quad a = -0.8 \quad and \quad -10 \leq n < 10$$

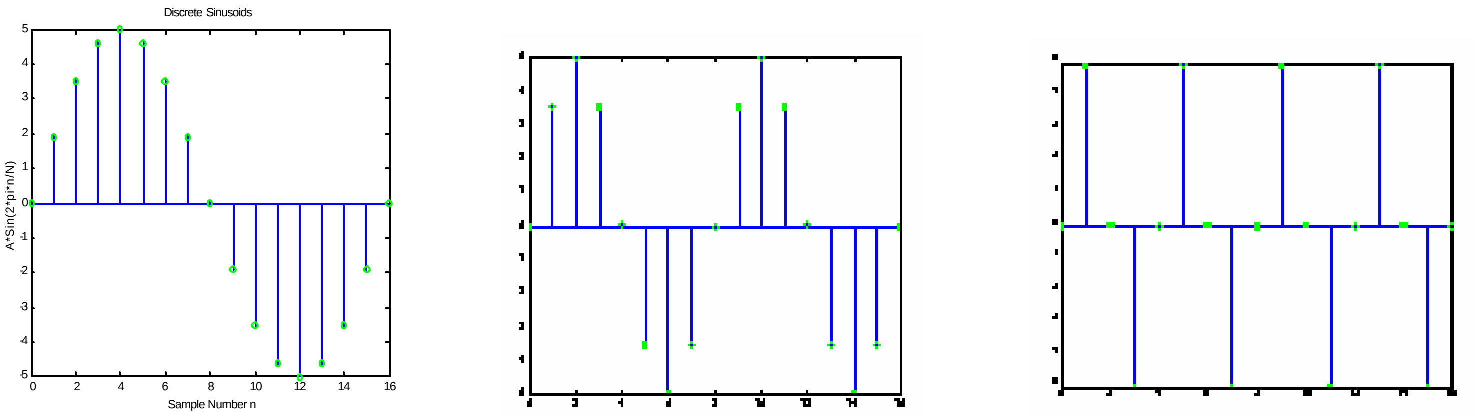
```
n=-10:1:10;
exp2 =10*(-.8).^n;
axis([-10 10 -30 30]);
stem(n,exp2);
xlabel('sample number, n'); ylabel ('alternating exp(n)');
title('Exponential sequence ')
grid; axis
```

2.3.5 Discrete Sinusoidal Signal:

$$x[n] = A.Sin(\Omega_0 n + j) = A.Sin(\frac{2p}{N} n + j) \qquad (2.22)$$

where $N = Period \quad and \quad \Omega_0 = Digital\ Fundamental\ Frequency.$

Example: 2.13: Effects of Sampling Rate on: $x[n] = A.Sin(2pn/16)$ where $N = 16\ Samples$.



Recall from trigonometry:

$$e^{jn(\Omega_0+2p)} = e^{j2pn}.e^{jn\Omega_0} = 1.e^{jn\Omega_0} = e^{jn\Omega_0} \quad (2.23)$$

We need to consider only one complete $2p$ interval, such as: $0 \le \Omega_0 < 2p$. This implies that for discrete trigonometric signals we will always be dealing with a finite interval.

Let us address the following question: Does the following equality hold everywhere:

$$e^{j\Omega_0(n+N)} \xrightarrow{?} e^{j\Omega_0 n} \quad (2.24)$$

The answer would be Yes if we have: $e^{j\Omega_0 N} = 1$

Let us use N: discrete fundamental period and m :a positive integer to yield N an integer:

$$\Omega_0 N = 2pm \quad \Rightarrow \quad N = m(2p/\Omega_0) \quad (2.25)$$

2.3.6 Sinusoidal Harmonics:

$$f_k[n] = e^{j\frac{2pk}{N}n} = e^{j\Omega_0 n} \quad (2.26a)$$

$$f_0[n] = e^{j0} = 1 \quad (2.26b)$$

$$f_1[n] = e^{j\frac{2p}{N}n} \quad (2.26c)$$

$$f_2[n] = e^{j\frac{2p}{N}2.n} \quad (2.26d)$$

$$f_{N-1}[n] = e^{j\frac{2p}{N}(N-1)n} \quad (2.26e)$$

$$f_N[n] = e^{j\frac{2p}{N}Nn} = e^{j2pn} = 1 = f_0[n] \quad (2.26f)$$

$$f_{N+k} = e^{j\frac{2p}{N}(N+k)n} = e^{j\frac{2p}{N}kn} = f_k[n] \quad (2.26g)$$

This last results implies that we only need to consider the harmonic numbers $\{0,1,\ldots,N-1\}$ since all other harmonics can be written in terms of these ***N distinct*** terms. This property of discrete signals is also known as the ***N-Point Resolution Property***.

2.3.7 Exponentially Modulated Sinusoidal Sequences:

$$x[n] = A.a^n.Cos(\frac{2p}{N}n + q) \quad (2.27)$$

where:

$A = Amplitude$

$a^n = Envelope\ of\ Modulated\ Signal$

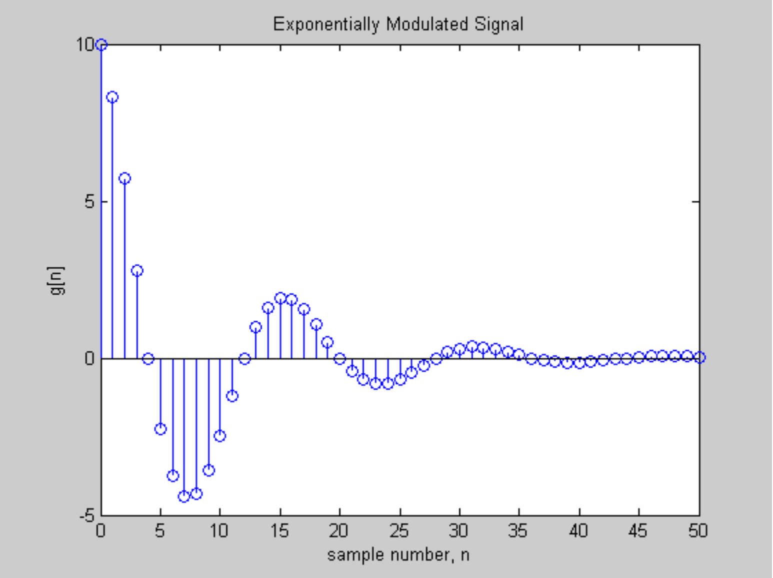
$Cos(\frac{2p}{N}n + q) = Oscillatory\ Component\ of\ Signal$

$q = Initial\ Phase\ when\ n = 0$

Example: 2.13: Exponentially modulating Sinusoidal Sequence

```
n=0:1:50;
exp=10*(0.9) .^n;
g=cos(2*n*pi/16);
h=exp .* g;
stem(n,h);
```

```
xlabel('sample number, n'); ylabel ('g[n]');
title('Exponentially Modulated Signal')
```



2.4 Some Terminology for Discrete Signals

Sequences:

$x[n]=\{\cdots,0,0,\bar{1},\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots\}$, where bar on top of "1" indicates origin of time; $n=0$.

Explicit mathematical representations:

$$x[n]=\begin{cases}0 & n<0\\ 2^{-n} & n\geq 0\end{cases} \qquad (2.28)$$

Recursive Equations:

$$x[n]=\begin{cases}0 & n<0\\ 1 & n=0\\ 0.5x[n-1] & n>0\end{cases} \qquad (2.29)$$

Energy Signals (Sequences): Discrete energy signals exhibit finite energy:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \tag{2.30}$$

Power Signals (Sequences): They have finite power:

$$P = Lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \tag{2.31}$$

Absolutely Summable Sequences:

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{2.32}$$

Periodic Signals (Sequences): $x[n]$ is periodic if, for some integer $N > 0$,

$$x[n+N] = x[n] \quad for\ all\ n \tag{2.33}$$

The smallest value of N that satisfies (2.33) is called the fundamental period and the fundamental frequency is a rational number given by:

$$\Omega_0 = m.\frac{2p}{N} \tag{2.34}$$

Example: 2.14: Find the fundamental periods of

$x[n] = Cos(\frac{7p}{9}n)$ and $y[n] = Cos(\frac{7}{9}n)$

for $x[n]$: $\Omega_0 = m.\frac{2p}{N} \Rightarrow \frac{\Omega_0}{2p} = \frac{m}{N} = \frac{7}{18} \Rightarrow m = 7\ and\ N = 18$

for $y[n]$: $\frac{\Omega_0}{2p} = \frac{m}{N} = \frac{7}{18p}$ Not a rational ratio and there is no fundamental period and the signal is an aperiodic sequence.

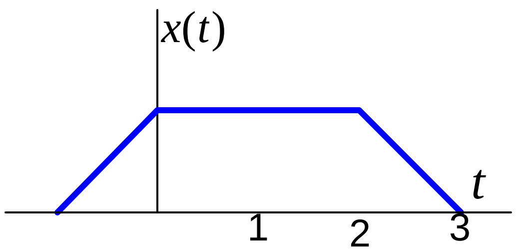
2.5 Derived Signals

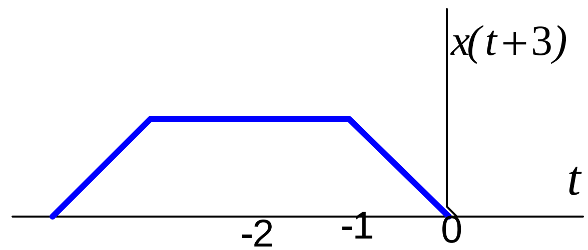
2.4.1 Shifting or Translating Signals:

- Assume *x(t)* is a triangular signal as shown below.
- Find $x(t-t_0)$ by delaying the original signal to the neighborhood of $t=t_0$.
- In the case of discrete signals *x(n)* can be shifted (delayed) to the neighborhood of $n=n_0$.

Example: 2.15. Given an *x(t)* as shown below obtain *x(t+3).*

$$x(t)=\begin{cases} t+1 & if \quad -1\le t\le 0 \\ 1 & if \quad 0\le t\le 2 \\ -t+3 & if \quad 2<t\le 3 \\ 0 & if \quad Otherwise \end{cases}$$





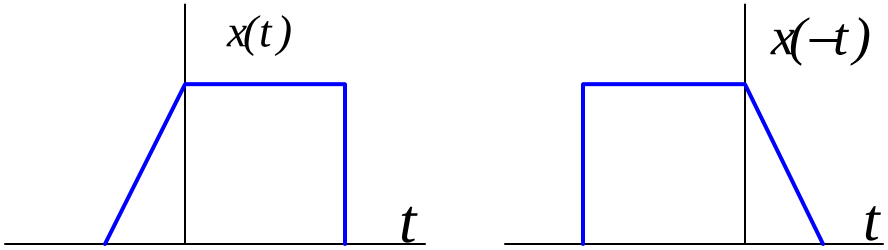
Let us replace t by $t+3$ everywhere in the above representation to obtain $x(t+3)$:

$$x(t+3)=\begin{cases} t+4 & if & -1\le t+3\le 0 \\ 1 & if & 0<t+3\le 2 \\ -t & if & 2<t+3\le 3 \\ 0 & if & Otherwise \end{cases}=\begin{cases} t+4 & if & -4\le t\le -3 \\ 1 & if & -3<t\le -1 \\ -t & if & -1<t\le 0 \\ 0 & if & Otherwise \end{cases}$$

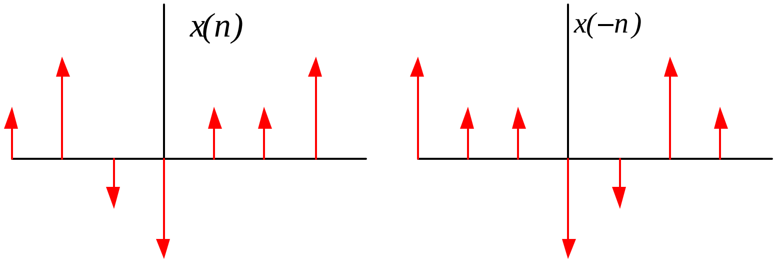
2.4.2 Reflection of Signals: Given an *x(t)* find *x(-t)* and equivalently, given an *x*[n] find *x*[-n].

Example: 2.16. Given an asymmetric analog signal find its reflection.

$$x(t)=\begin{cases} t+1 & if \quad -1<t\le 0 \\ 1 & if \quad 0<t\le 2 \\ 0 & Otherwise \end{cases} \qquad x(-t)=\begin{cases} -t+1 & if \quad -1<-t\le 0 \\ 1 & if \quad 0<-t\le 2 \\ 0 & Otherwise \end{cases}=\begin{cases} -t+1 & if \quad 1>t\ge 0 \\ 1 & if \quad 0>t\ge -2 \\ 0 & Otherwise \end{cases}$$



Example: 2.17. Given an asymmetric discrete signal find its reflection.

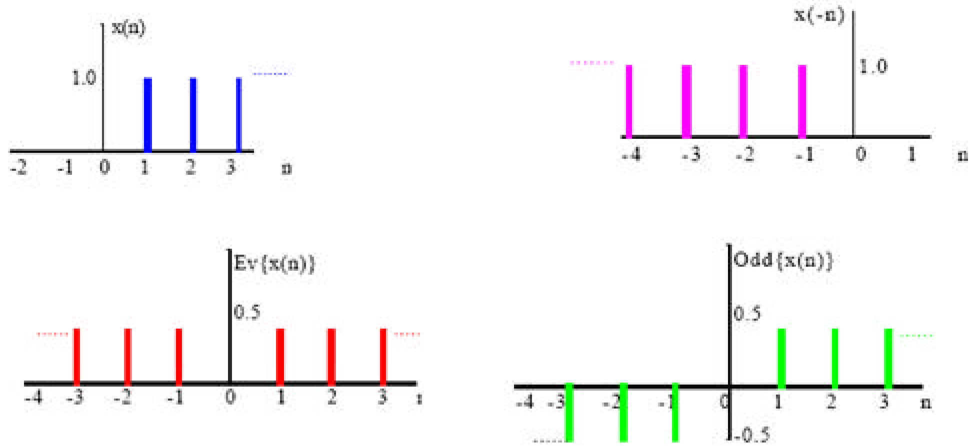


2.4.3 Even and Odd Portions of a Signal:

Continuous Case: $Ev\{x(t)\}=\frac{1}{2}[x(t)+x(-t)]$ and $Odd\{x(t)\}=\frac{1}{2}[x(t)-x(-t)]$ (2.35a)

Discrete Case: $Ex\{x[n]\}=\frac{1}{2}(x[n]+x[-n])$ and $Odd\{x(n)\}=\frac{1}{2}(x[n]-x[-n])$ (2.35b)

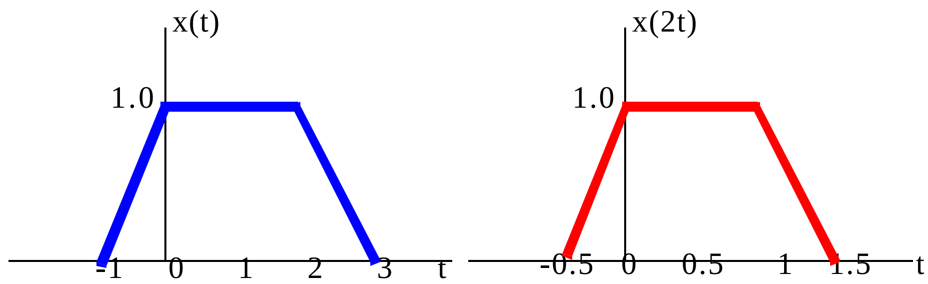
Example: 2.18. Given a discrete signal find its even and odd portions:



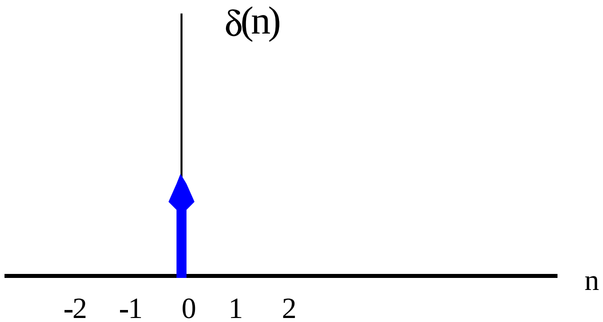
2.4.4.Time-Scaling: Given a continuous or discrete time signal, we can scale it to $x(kt)$ or $x[kn]$ by replacing *t* or *n* by kt or kn everywhere the function is defined.

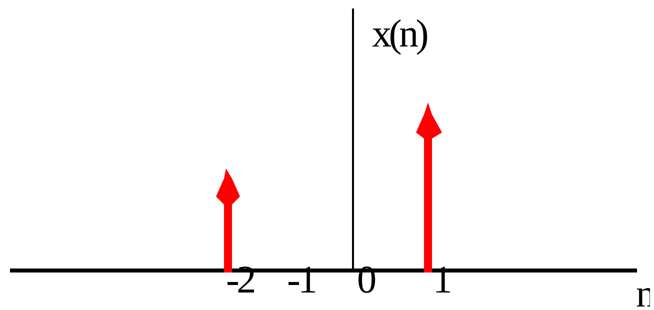
Example: 2.19. Given the following signal find x(2t). Let us assume that $\hat{t}$ stands for the new time-variable of the system. Then everywhere we have *t* we replace it with *2t*.

$$x(t)=\begin{cases} t+1 & if \quad -1\le t\le 0 \\ 1 & if \quad 0<t\le 2 \\ -t+3 & if \quad 2<t\le 3 \\ 0 & Otherwise \end{cases} \Longrightarrow x(2t)=\begin{cases} 2t+1 & if \quad -1\le 2t\le 0 \\ 1 & if \quad 0<2t\le 2 \\ -2t+3 & if \quad 2<2t\le 3 \\ 0 & Otherwise \end{cases} = \begin{cases} 2\hat{t}+1 & if \quad -0.5\le \hat{t}\le 0 \\ 1 & if \quad 0<\hat{t}\le 1 \\ -2\hat{t}+3 & if \quad 1<\hat{t}\le 1.5 \\ 0 & Otherwise \end{cases} \qquad (2.36)$$



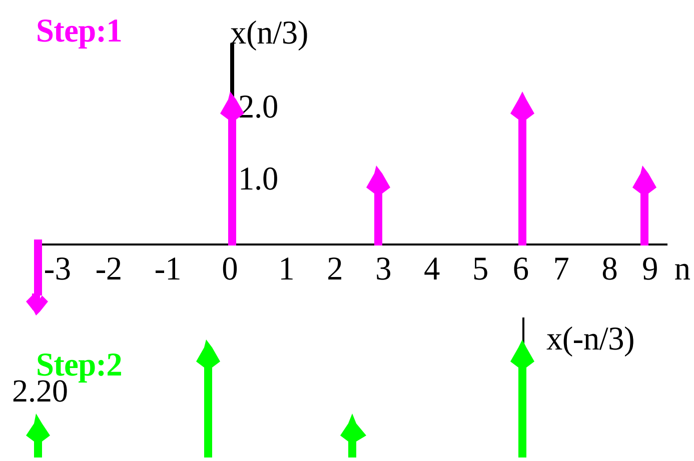
Example: 2.20. Given a delta function find $x[n] = 2d[1-n] + d[n+2]$

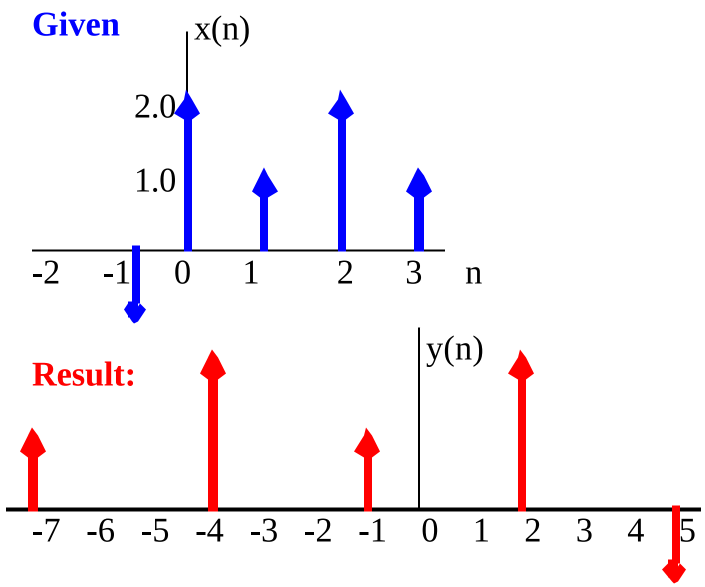




Example: 2.21. Given the following discrete function find:

$$y[n] = x[-\frac{n}{3} + \frac{2}{3}] = x[-\frac{n-2}{3}]$$





(2.33)

2.6 Properties of Continuous and Discrete-Time Systems

We will next investigate the properties of systems for continuous and fully discrete systems (paths 1 and 2 in Figure 1.3.)

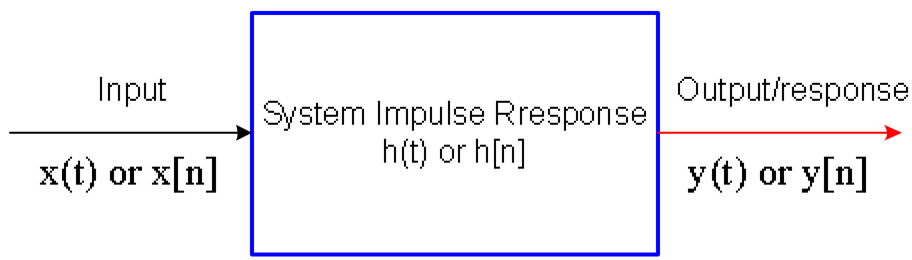


Figure 2.1 Unified System Impulse Response Model

2.6.1 Linearity:

A system is linear if it satisfies both of the following two conditions:

a. Zero input must yield zero output. (Zero-in/Zero-out Property)

b. Principle of superposition holds.

$$a.x_1(t) + b.x_2(t) \quad \Rightarrow \quad a.y_1(t) + b.y_2(t) \qquad \text{Continuous} \qquad (2.37a)$$

$$a.x_1[t] + b.[_2(t] \quad \Rightarrow \quad a.y_1[t] + b.y_2[t] \qquad \text{Discrete} \qquad (2.37b)$$

Example 2.22 (Discrete): Ideal accumulator, as in the case of a discrete current charging a capacitor.

$$y[t] = K\sum_{n=0}^{N} x[n]$$

Task: Check if this system is linear?

Zero input condition: Let $x[n] = 0 \quad for\ all\ n$. Then summing zeros yields a zero, condition a is O.K.

b. Superposition:

If the input is: $x[t] = a.x_1[t] + b.x_2[t]$ then the output will be:

$$y[n] = K\sum_{n=0}^{N} x[n] = K.(a\sum_{n=0}^{N} x_1[n] + b\sum_{n=0}^{N} x_2[n]) = aK\sum_{n=0}^{N} x_1[n] + bK\sum_{n=0}^{N} x_2[n] = a.y_1[n] + by_2[n]$$

Thus the system is linear.

Example 2.23 (Continuous): Consider the following input/output behavior of a discrete exponential system:

$$y[n] = e^{x[n]}$$

Zero input condition: Let $x[n] = 0 \quad for\ all\ n$. Then $\quad y[n] = e^0 = 1$

and the first condition is violated. Therefore, there is no need to check the second condition and the system is non-linear.

2.6.2 Time-Invariance: A system is time-invariant if a time-shift to t_0 in the input $x(t)$ causes the time-shift t_0 in the output $y(t)$. Similar for the discrete case with n replacing t.

Example 2.24: Check if a sinusoidal system is time-invariant

$$y(t) = Cos(x(t))$$

Procedure for testing time-invariance:

a. Let $x_1(t)$ be the input, which will yield an output: $y_1(t) = Cos(x_1(t))$.

b. Consider a delayed version of the input:

$$x_2(t) = x_1(t - t_0).$$

It results in an output:

$$y_2(t) = Cos(x_2(t)) = Cos(x_1(t - t_0))$$

c. Let us use the input in b:

$$y_2(t) = y_1(t - t_0) = Cos(x_1(t - t_0))$$

Since these two results are identical the system is time-invariant.

Example 2.25: Check if a moving average system (discrete Simpson's rule) is time-invariant.

$$y[n] = \frac{1}{N + M + 1} \sum_{k=-M}^{N} x[n - k] \qquad (2.38)$$

If we replace n with $n - n_0$ in the input then every term on the right-hand side of (2.38) will be shifted by n_0 resulting in a shift of n_0 in $y[n]$ to produce $y[n - n_0]$. Hence the moving average system is time-invariant.

2.6.3 Memoryless Systems and Systems with Memory: A system is memoryless if the input $x(t)$ or $x[n]$, the output $y(t)$ or $y[n]$, and the system parameters all depend only on the same time instant t or n and no other time. We have the following input-output relationship in functional form: $y(t) = F\{x(t)\}$ or $y[n] = F(x[n])$. Memory content of linear systems exhibit one of the two forms:

- Time Variant Linear systems:

$$y(t) = k(t).x(t) \qquad y[n] = k[n].x[n] \qquad (2.39)$$

Here $k(t)$, $k[n]$ stands for a time-dependent coefficient.

- Time-Invariant Linear Systems:

$$y(t) = K.x(t) \qquad y[n] = K.x[n] \qquad (2.40)$$

Here K is a constant-valued coefficient as in the case of a resistor. $K = R\ Ohms$.

Example 2.26: Consider the Ohm's Laws across a resistor and a capacitor.

$$v(t) = R.i(t)\,Volts \tag{2.41}$$

and

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t).dt \tag{2.42}$$

Clearly, the first one is a memoryless system, whereas, the second one has memory since charges accumulate in the capacitor *C*.

2.6.4 Causal Systems: A system is causal, or non-anticipatory, or physically realizable, if *y(t)* at any time t_0 depends on values of the input for $t \le t_0$. In other words, no future inputs are used in computing *y(t).* Similarly, a discrete system is causal if $y[n]$ does not depend on future values of $x[n]$.

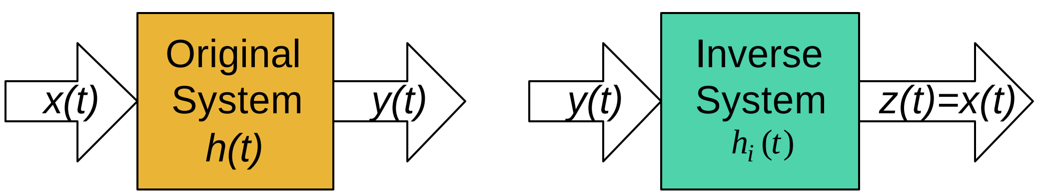
Example 2.27: Check if the following two averaging systems are causal.

a. $y(t) = \frac{1}{2}[x(t - t_0) + x(t)]$ For $t \ge 0$ this system will be causal.

b. $y[n] = \frac{1}{3}(x[n - n_0] + x[n] + x[n + n_0])$ (Three-point moving averager)

For all values of $n_0 \neq 0$, the output $y[n]$ will need input information from a future time instant as well as the current time and a past instant. Therefore, this simple averaging mechanism is a non-causal system, which is also known as the three-point smoother, or an integrator.

2.6.5 Invertible Systems: A system is invertible if there is an inverse of the system to recover the original input *x(t)*.



Example 2.28: Given: $y(t) = K.x(t) \quad and\ K = finite\ number$

We compute: $1/K$ to have the numerical inverse. We can go through these two systems one after the other to recover the original input exactly.

$$z(t)=\frac{1}{K}.y(t)=\frac{1}{K}.K.x(t)=x(t)$$

If the input-output relationship in the original system is a square-law type, we have:

$$y[n]=x^2[n]$$

$$z[n]=\sqrt{y[n]}=\sqrt{x^2[n]}=x[n]$$

But if we submit negative of the first input to the same system-pair we obtain:

$$y[n]=(-x[n])^2=x^2[n]$$

$$z[n]=\sqrt{(-x[n])^2}=\sqrt{x^2[n]}=x[n] \quad not \quad -x[n]$$

It is clear that we did not recover the original signal $-x[n]$, but its negative. So, a simple square-law and square-root law type relationships are not invertible.

2.6.6 BIBO Stability of Systems: A system Bounded-Input Bounded Output (BIBO) Stable if a bounded input (always finite) yields another bounded function as its output. The following two conditions constitute the procedure for checking this property.

(a) Input signal $x(t)$ is bounded if $|x(t)|<B_1<\infty \quad for\ all\ values\ of\ t.$

(b) Resulting output $y(t)$ is bounded if $|y(t)|<B_2<\infty \quad for\ all\ values\ of\ t.$

Similarly for discrete case replace t with n.

Example 2.29:

Given a system : $y[n]=a^{x[n]}$ and

An input: a unit-step signal $x[n]=u[n]$.

(a) First condition: Is the input signal bounded?

Since $|x[n]|=|u[n]|\leq 1$ we have an upper bound value of "1".

(b) Second condition: Does this input produce a bounded output?

$$|y[n]|=\left|a^{x[n]}\right|=a^{|u[n]|}\leq a^1=a$$

From these two finite or bounded results we conclude that this exponential-law device is BIBO stable for step input signals.

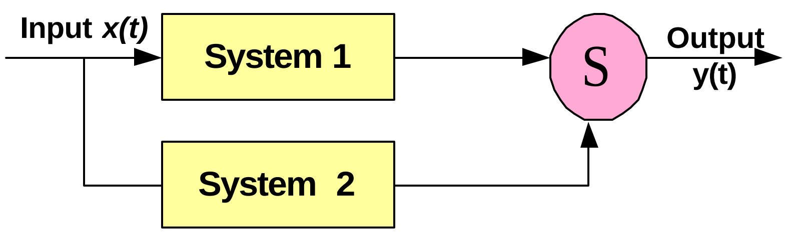
Example 2.30 : Let us analyze a counter-example to this. Let the system be an ideal capacitor *C* and find the response of this device to a step-input current *i(t)*.

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$

$$|v(t)| = \left|\frac{1}{C}\int_{-\infty}^{t} i(t)dt\right| = \frac{1}{C}\left|\int_{-\infty}^{t} u(t)dt\right| = \begin{cases} \frac{1}{C}\int_{-\infty}^{t} u(t)dt = \frac{1}{C}\int_{0}^{t} 1.dt & if\ t > 0 \\ 0 & if\ t \le 0 \end{cases} = \begin{cases} \frac{t}{C} \Rightarrow \infty & if\ t > 0 \\ 0 & if\ t \le 0 \end{cases}.$$

From this result we conclude that the output *y(t)* is not bounded and the output goes to infinity for large values t. However, it is finite for small values of t. Nevertheless, the system is NOT BIBO stable.

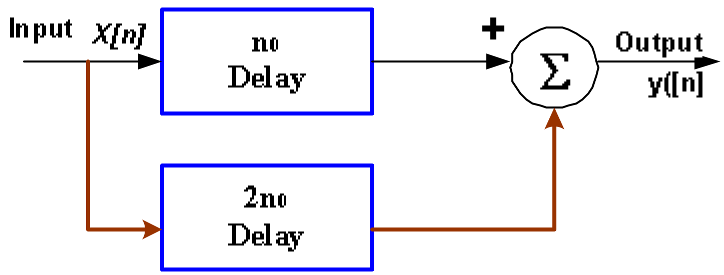
2.6.7 Feed-forward (Open-Loop) Systems: A system is called a feed-forward or open-loop system if the information path is from the input port towards the output at all times. It is not difficult to see that in computing the output for these systems we need to know only the present and past input values. In other words, we do not need past information nor the past output values. The general block diagram consists of sub-systems and an adder as shown above.



The system behavior is described by: $y(t)\ or\ y[n] = System_1\ \text{Re}\,sponse + System_2\ \text{Re}\,sponse$ (2.43)

Example 2.31: Consider the following difference equation:

$$y[n] = x[n - n_0] + x[n - 2n_0] \quad (2.44)$$

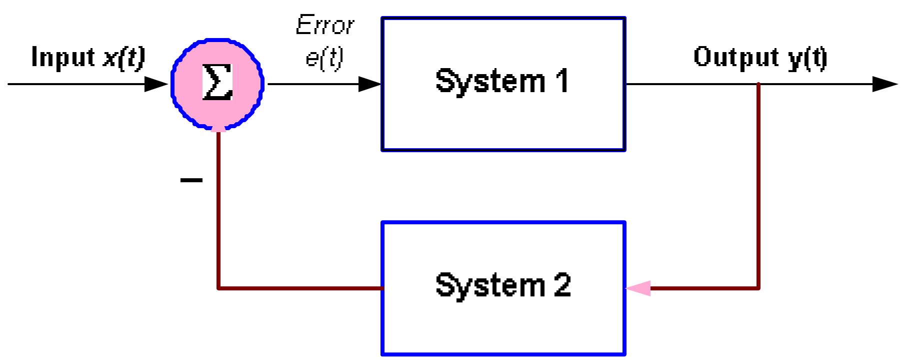


2.6.8 Feedback (Closed-Loop) Systems: A system is called a feedback or closed-loop system if the information path contains a feedback path from output towards the input side. These systems generally depend on past outputs as well as current and past inputs. They are governed by a pair of equations:

$$y(t) = H_1(e(t)) \quad and \quad e(t) = x(t) - H_2(y(t)) \qquad (2.45a)$$

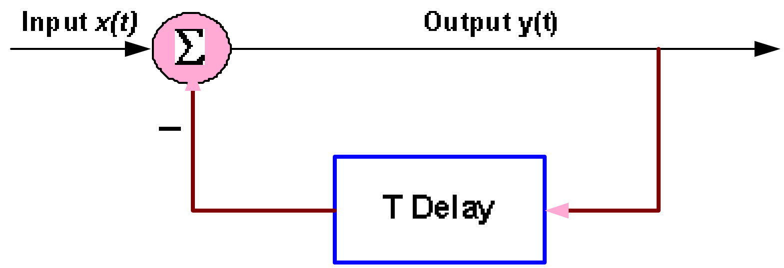
$$y[n] = H_1(e[n]) \quad and \quad e[n] = x[n] - H_2(y[n]) \qquad (2.45b)$$

Here $H_1(\bullet) \quad and \quad H_2(\bullet)$ are the governing system equations for these systems.



Example 2.10: Consider the following difference equation:

$$y(t) = x(t) - y(t - T)$$



This is a single-tap delay feedback system very frequently used in linear prediction for speech/image and in motion estimation for video processing applications.

2.6.9 FIR and IIR Systems: A discrete Linear Time-Invariant (LTI) system has finite impulse response (FIR) if we can find integers $N_1 \le N_2$ such that

$$h[n] = 0 \quad when \quad n < N_1 \ and \ n > N_2. \tag{2.46}$$

Otherwise, the LTI system has an infinite impulse response (IIR). Similarly, we can talk about FIR and IIR system responses for the continuous case.

Example 2.11: A rectangular time-window with length N units has a FIR impulse response:

$$h[n] = u[n] - u[n-N] \tag{2.47}$$

But the LTI system with

$$h[n] = a^n.u[n] = \begin{cases} a^n & n \ge 0 \\ 0 & Otherwise \end{cases} \tag{2.48}$$

is IIR. Because we cannot find an N_2, which is finite to stop at the end of the response.

Important Observation and Design Decision Issue: FIR systems are absolutely and unconditionally stable and hence, they are the designers choice for implementations if the computational cost does not become an issue.

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=N_1}^{N_2} |h[n]| < \infty \tag{2.49}$$