Chapter 1 – Probability Models

Mathematical Models

- **Experiments:** Costly way of testing a design or solve a problem.
- **Model:** Approximate representation of a physical situation.
- **Useful Model:** Able to explain all relevant aspects of a given phenomenon.
- **Mathematical Models:** If observational phenomenon has measurable properties then a mathematical model consisting of a set of assumptions about the system is employed.

Conditions under which an experiment is performed and a model is assumed are very critical. Change the assumptions then a “good” model can be a great fiasco.

![Model Development Flowchart](image)

- Computer Simulation Models: They mimic or simulate the dynamics of a system
- Deterministic Models: Lab and textbook cases, conditions determine outcome
  1. Circuit Theory
  2. Ohm’s Law
  3. Kirchoff’s Laws
  4. Transforms: FFT; Laplace Transforms
  5. Convolution: Input/output behavior of systems with well-defined coefficients
- Probabilistic (Stochastic, Random) Models: involve phenomena that exhibit unpredictable variation and randomness.

**Ex:** Urn with three balls; (0,1,2 marked)
- Outcome: A number from the set \{0,1,2\}
- Sample Space: All possible outcomes of an experiment: \(S = \{0,1,2\}\)

![Graph showing sample outcomes](image)

**Statistical Regularity:**

Relative Frequency:

\[ f_k(n) = \frac{N_k(n)}{n} \]

- \(n\): # of experiments under identical conditions;
- \(k\): Counter index
  - \(N_0\): # of “0” balls
  - \(N_1\): # of “1” balls
  - \(N_2\): # of “2” balls

\(\text{in } n \text{ tries}\)

Let \(n \to \infty\) then \(\lim_{n \to \infty} f_k(n) = P_k\); where \(P_k\): probability of the outcome \(E_k\).
Properties of Relative Frequency:
Suppose a random experiment has \( K \) possible outcomes: \( S = \{1,2,\ldots,K\} \). Then in “\( n \)” trials we have

\[ 0 \leq N_k(n) \leq n \quad \text{for} \quad k = 1,2,\ldots,K \quad \Rightarrow \quad 0 \leq f_k(n) \leq 1 \]

and

\[ \sum_{k=1}^{N} N_k(n) = n \quad \text{which implies:} \quad \sum_{k=1}^{N} f_k(n) = 1 \]

Event \( \equiv \) Any outcome of an experiment satisfying certain condition(s).

Ex: Consider the 3-ball urn experiment

A: even = \{0,2\} then,
\[
f_A(n) = \frac{N_0(n) + N_2(n)}{n} = f_0(n) + f_2(n)
\]

Disjoint (mutually exclusive) events: If A or B can occur but not both, then
\[
f_C(n) = f_A(n) + f_B(n)
\]

Relative frequency of two disjoint events is the sum of their individual relative frequency.

**Kolmogorov’s axioms to form a Theory of Probability: Assumptions:**
1. Random experiment has been defined and the sample space S has been identified.
2. A class of subsets of S has been specified.
3. Each event A has been assigned a number P(A) such that,
   1. \(0 \leq P(A) \leq 1\)
   2. \(P(S) = 1\)
   3. If A and B are mutually exclusive events then
      \(P(A \text{ or } B) = P(A) + P(B)\)

Kolmogorov’s axioms are sufficient to build a consistent Theory of Probability.

**Example:** Packet Voice Communication system Efficiency
Due to silences voice communication is very inefficient on dedicated lines. It is observed that only “1/3” of the time actual speech goes through. How to increase this rate by using prob. approaches???

**Solution:** Error vs rate trade off in digital information (BCS) transmission/storage

\[
\begin{align*}
0 & \rightarrow 0 \quad 0 \quad 0 \\
1 & \rightarrow 1 \quad 1 \quad 1
\end{align*}
\]

+ Majority Rule to make decision at the receiver.

If each bit has a \(P_e = 10^{-3}\) due to this simple scheme, \(P_e \rightarrow 3 \times 10^{-6}\).

**Cost:** Rate is decrease to 1/3 of the original.

Binary Symmetric Channel with cross-over probability: \(\varepsilon = P_e = 10^{-3}\)
Example: Signal Enhancement Using Filters
Given a signal $x(t)$ corrupt with noise and has a Signal-to-Noise Ratio value SNR. If you filter this noisy signal with a properly designed—hopefully adaptive, filter to suppress noise, we obtain an enhanced signal, (smoothed by the filter.)

Example: Multi User Systems with Queues: Resource sharing

Two performance curves for multi-user computer queuing studies:
Example: System Reliability: Cascade vs. Parallel Systems

![Diagram of series and parallel configurations of components.]

**Issues:** Need of a clock vs. the system delay or throughput rate.

**Ex: Prob. 1.1**

**Experiment:** Selecting two balls in succession from an urn with 2 black + 1 white ball without replacement.

a) **Sample Space:** $S = \{bb, bw, wb\}$  
   - If: 1st ball is B, then 2nd is B or W  
   - If: 1st ball is W, then 2nd is only B

b) **With replacement:** All outcomes are W or B, 2 then  
   $S = \{bb, bw, wb, ww\}$

c) $f_{ww}(n)$ in part (a)?
   - Not possible to have W then W $\Rightarrow f_{ww}(n) = 0$

d) $f_{ww}(n)$ in part (b)?
   - $N_w(n) = 1/3$, since 1 W and 2 B balls always
     Of these 1/3 outcomes again 1/3 are W since balls are replaced.

   $\therefore f_{ww}(n) = (1/3)*(1/3) = 1/9$

2nd draw is effected by the outcome of 1st draw?
   a) Yes
   b) No.