Chapter: 3

COMMUNICATION IN NOISELESS AND NOISY CHANNELS

3.1 System Response and Architectures

From the results of the convolution theorem, we know that the response of a linear time-invariant system (LTI) in time and frequency domains are given by:

$$y(t) = x(t) * h(t) \tag{3.1a}$$

$$Y(f) = X(f).H(f) \tag{3.1b}$$

where $h(t)$ and $H(f)$ are the impulse-response and the frequency-response of the system in question. In systems engineering community, we use the amplitude and phase representation for (3.1b):

$$|Y(f)| = |X(f)|.|H(f)| \quad \text{and} \quad \arg Y(f) = \arg X(f) + \arg H(f) \tag{3.2}$$

and the spectral density and the total energy are given by:

$$|Y(f)|^2 = |X(f)|^2 .|H(f)|^2 \tag{3.3a}$$

$$E_y = \int_{-\infty}^{\infty} |X(f)|^2 .|H(f)|^2 .df \tag{3.3b}$$

Example 3.1[1]**:** Assume that a distortionless channel can be modeled as the impulse response of an RC circuit and further assume that x(t) and y(t) are the input and outputs of this channel. $|H(f)|$ $t_d(w)$

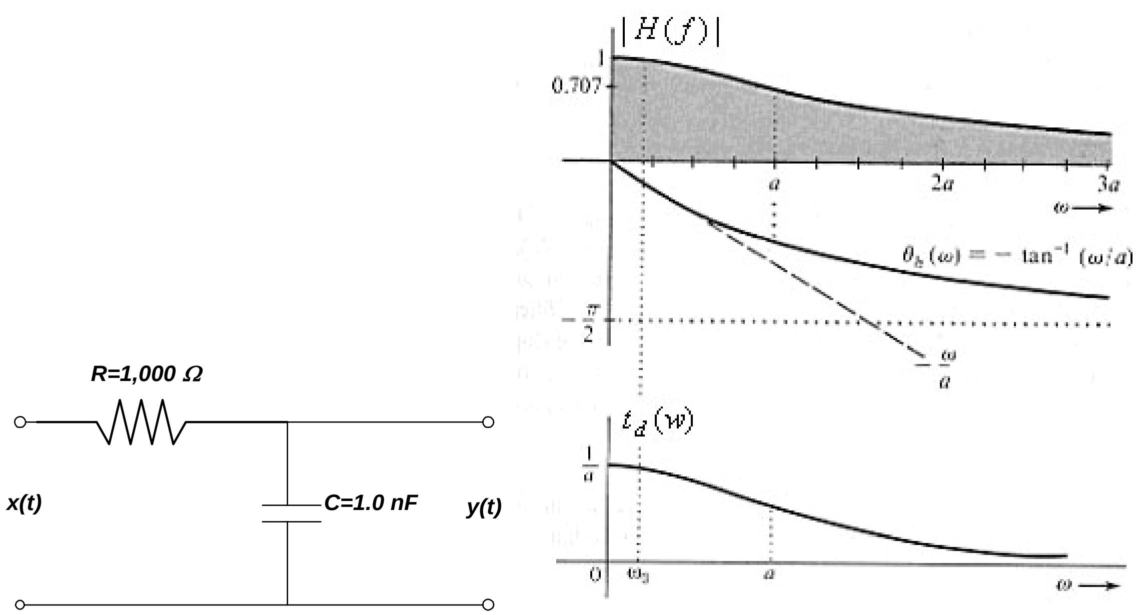

Figure 3.1 Amplitude and Phase responses of an An RC circuit.

The frequency response of this circuit is easily found from the Voltage Division Law in the frequency-domain:

$$H(w) \equiv \frac{Y(w)}{X(w)} = \frac{1/jwC}{R + 1/jwC} = \frac{1}{1 + jwRC} = \frac{a}{a + jw} \tag{3.4}$$

where

$$a \equiv 1/RC = 1/(10^3 * 10^{-9}) = 10^6 .$$

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[1] *Modern Analog and Digital Communication Systems,* Third Edition, B.P. Lathi, Oxford University Press, 1998

However, for small frequencies in the range: $w << a = 10^6$, the magnitude response and the phase response can be approximated with:

$$|H(w)| = \frac{a}{\sqrt{a^2 + w^2}} \cong 1 \qquad \text{and} \qquad \theta_h(w) = -\arctan(\frac{w}{a}) \cong -\frac{w}{a} \tag{3.5}$$

which are also plotted in Figure 3.1.

Let us study these plots carefully:

- From the magnitude spectrum, we see that if $w \leq 200{,}000$ radians/second then the magnitude response deviates from the peak value of "1.0" within 2%.
- The phase response deviates from the linear behavior by 1.5%.

Using the approximations for quantities in (3.5) in our range of interest, we obtain a group delay figure, which is defined as the derivative of the phase response with respect to *w*:

$$\theta_h(w) \cong -\frac{w}{a} = -\frac{w}{10^6} \qquad \text{and} \qquad t_g = \frac{d\theta_h(w)}{dw} \cong \frac{1}{a} = RC = 10^{-6} = 1.0\ \mu s \tag{3.6}$$

3-dB bandwidth: *B* is defined the frequency at which the magnitude drops to 0.707 of its normalized peak value, or equivalently, the power across $1.0\,\Omega$ resistor drops to ½.

3-db bandwidth for this example is computed from:

$$0.707 = \frac{10^6}{\sqrt{10^{12} + B^2}} \quad \Rightarrow \quad B = 159.23\, kHz \tag{3.7}$$

Next, if we assume that the input signal is a 100 Hz sinusoid: $\Rightarrow \quad x(t) = A.Cos(200\pi t)$.

For this input signal we have: $w_1 = 200\pi << a = 10^6$ and we can easily use the approximation:

$$|H(w)| \cong 1$$

and substituting the values for $|H(w)|$ and t_g, the output equation in time-domain is written as:

$$\begin{aligned} y(t) &= A.|H(w)|.Cos[200\pi(t - t_g)] \approx A.1.Cos(200\pi t - 200\pi.10^{-6}) \\ &= A.Cos(2\pi t - 2\pi.10^{-4}) \end{aligned} \tag{3.8}$$

Communication systems are normally built from some specific arrangement of smaller units designed to perform a particular task. These units are combined into a larger system by a parallel, cascade, feed-forward, feedback and various combinations thereof. In Figure 3.1 we show three different setups

Operation	Time-Domain	Transfer Function
Scalar Multiplication	$y(t) = \pm K.x(t)$	$H(f) = \pm K$
Differentiation	$y(t) = dx(t)/dt$	$H(f) = j2\pi.f$
Integration	$y(t) = \int_{-\infty}^{t} x(u)du$	$H(f) = 1/\,j2\pi.f$
Time-Delay (shift)	$y(t) = x(t-\tau)$	$H(f) = e^{-j2\pi/\tau}$

Parallel (feedforward) and cascade connections of systems:

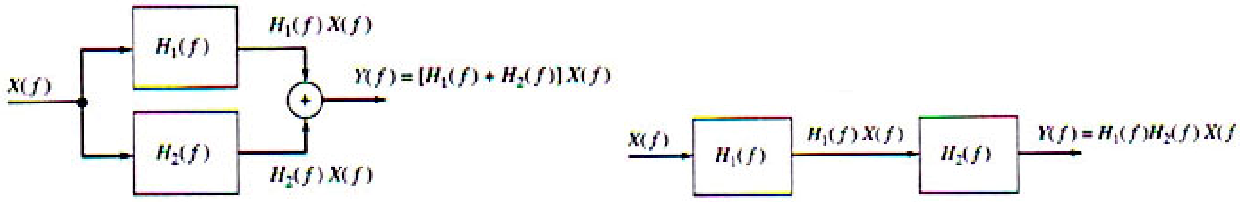


Figure 3.2 Parallel (feedforward) and cascade connection of two subsystems

Parallel connection: $H(f) = H_1(f) + H_2(f)$ and $h(t) = h_1(t) + h_2(t)$ (3.9)

Cascade connection: $H(f) = H_1(f).H_2(f)$ and $h(t) = h_1(t) * h_2(t)$ (3.10)

Feedback connections of Systems:

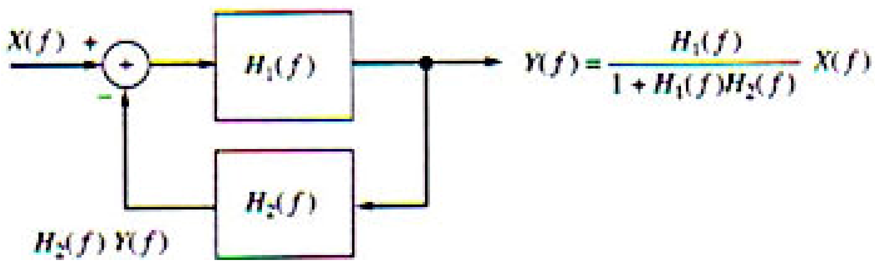


Figure 3.3 Feedback (negative) connection of two subsystems

Feedback (negative) connection:

$$Y(f) = H_1(f).[X(f) - H_2(f).Y(f)] = \frac{H_1(f)}{1 + H_1(f).H_2(f)}.X(f)$$

$$H(f) = \frac{H_1(f)}{1 + H_1(f).H_2(f)} \quad (3.11)$$

3.2 Conditions for Distortionless Communication

In order to study the information transmission properly we need to study the conditions for lossless communication in both ideal channels and real-life channels, which includes the study of Gaussian noise, use of filters as channel models and other for degradations.

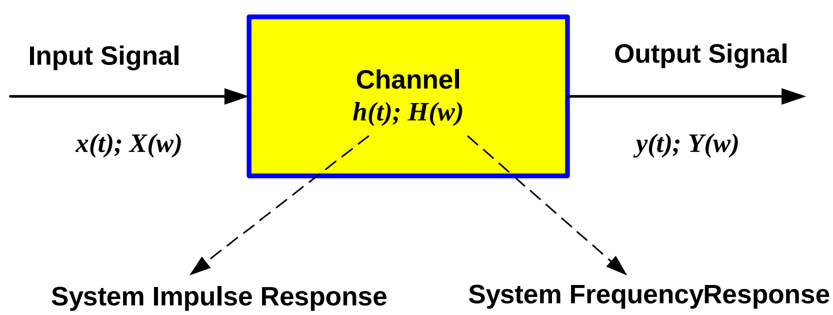


Figure 3.4 System Response Block Diagram

Distortionless transmission over a linear system is achieved by convolving an input signal $x(t)$ with the impulse response *h(t)* of the linear system in question.

$$y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (3.12)$$

$$Y(w) = X(w).H(w) = |X(w)|.|H(w)|.e^{j(\theta_x+\theta_h)} \quad (3.13)$$

From these expressions it is clear that the signal can be transmitted without any distortion if and only if $x(t)$ and $y(t)$ have identical wave shapes within:

- a multiplicative constant and
- a time-delay (linear phase-shift).

In other words, we should be able to write:

$$y(t) = k.x(t - t_d) \quad (3.14)$$

In order to meet this constraint we must have the following two conditions satisfied:

1. The magnitude of the system impulse response must be flat, at least, over the band of interest:
   $|H(w)| = k;$ a real-valued constant (3.15)
2. All frequency components must reach the output side with the same time delay t_d.

Let us elaborate these important conditions on lossless transmission by studying the case where $Cos(wt)$ is the input and t_d is the constant time-delay involved. For this situation, we can write:

$$Cos[w(t - t_d)] = Cos(wt - wt_d) = Cos(wt + \theta_h(w)) \quad (3.16)$$

which is equivalent to:

$$\theta_d(w) = -t_d.w \quad (3.17)$$

Clearly, (3.17) is a linear function of angular frequency w. In other words; the phase response has a constant slope $(-t_d)$ as shown in Figure 3.5. Furthermore, the frequency response of the output signal, $Y(w)$, is given by:

$$Y(w) = X(w).H(w) = X(w).k.e^{-jwt_d} \quad (3.18)$$

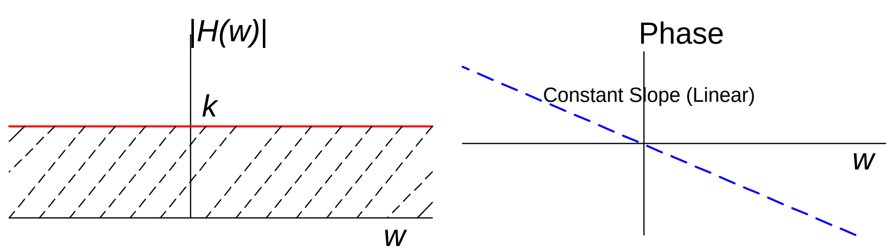


Figure 3.5 Magnitude and Phase Characteristics of a distortionless channel.

Let us re-examine the final result in Example 3.1, i.e. (3.8):

$$y(t) = A.|H(w)|.Cos[200\pi(t - t_g)] \approx A.1.Cos(200\pi t - 200\pi.10^{-6})$$

$$= A.Cos(2\pi t - 2\pi.10^{-4})$$

We now conclude that the transmission is distortionless and the output will be a $10^{-6}\ s = 1.0\ \mu s$ delayed replica of the input sinusoid.

In summary:

1. We have amplitude distortion when $|H(f)| \neq |K|$, which is normally observed in all real-life systems (varying with frequency as in Figure 3.2-2 of Carlson) but it is not very critical as long as the tapering-off does not take place inside the useful frequency range critical for the application.

2. We have phase or delay distortion when $\arg H(f) \neq -2\pi . t_d f \pm m180^o$ and the corresponding time-delay is found from:

$$t_d(f) = -\frac{\arg H(f)}{2\pi f} \tag{3.19}$$

It is worth noting that there is usually confusion between constant time-delay and constant phase-shift. The first is desirable and required for distortionless transmission. But the latter causes distortion in general and it is expressed in terms of group delay:

$$t_g = -\frac{1}{2\pi}\frac{d\theta(f)}{df} \tag{3.20}$$

where $\theta(f) = \arg H(f)$. If the system function is synchronous with the signal, .i.e., $\phi_0 = 0$ then group delay and the phase delay are the same: $t_g = t_d$.

3. Non-linear distortion occurs when the system includes non-linear elements (flip-flops, transistors, switches, etc).

3.3 Sources of Distortion in Communication Channels

If the frequency response of a communication channel $H(w)$ is not constant over a band of frequencies of interest, then frequency components of the input signal $X(w)$ is shaped differently due to the particular characteristics of the transmission medium, i.e., the channel. Effects of noise and other disturbances in the channel have been studied in order to understand the performance of different communication systems. Furthermore, these components might be delayed in a time-varying fashion due to non-constant delays (non-linear phase response) in the channel. As a consequence, it is very common to observe "Pulse Spreading, Dispersion" in digital communication systems. In particular, systems using time-division multiplexing (TDM) method to combine a number of information channels for efficient transmission are severely affected by pulse spreading. Let us study some of these effects with examples.

Example 3.2: Suppose the communication channel has the following frequency response:

$$H(w) = \begin{cases} (1 + k.Cos(wT).e^{-jwt_d} & if \ |w| \leq 2\pi B \\ 0 & otherwise \end{cases} \tag{3.21}$$

It is worth noting that the amplitude is shaped with a raised-cosine (1+*k*) form and the phase is linear function of *w* with slope t_d. Here T is the period of the raised-cosine function. Suppose that a generic signal $x(t)$ is transmitted through this channel. The task is to find the response of the channel, which is given by the convolution of the input with the impulse response of the channel:

$$y(t) = x(t) * h(t)$$

equivalently, in frequency-domain:

$$\begin{aligned} Y(\omega) &= X(\omega)H(\omega) = X(\omega)(1 + k\cos T\omega)e^{-j\omega t_d} \\ &= X(\omega)e^{-j\omega t_d} + k[X(\omega)\cos T\omega]e^{-j\omega t_d} \end{aligned} \tag{3.22}$$

The output in time-domain is simply the inverse Fourier transform of (3.22):

$$y(t) = F^{-1}\{Y(\omega)\} = x(t - t_d) + \frac{k}{2}[x(t - t_d - T) + x(t - t_d + T)] \tag{3.23}$$

As it is clear from the above expression, the output signal is composed of the original signal delayed by t_d plus its two echoes (reflections) at times $\mp T$ later/earlier, respectively.

Distortion from Channel Non-Linearities:

Suppose that our system exhibit non-linear characteristics then the output will be a non-linear function of the input signal:

$$y(t) = f(x(t))$$

where, *f*(.) stands for the non-linear functional form. Let us expand this non-linear function in terms of power series:

$$y(t) = a_0 + a_1.x(t) + a_2.x^2(t) + \cdots a_k.x^k(t) + \cdots \quad (3.24)$$

Recalling the Fourier transform property of the multiplication of $x(t)$ with another time-function, --again $x(t)$ in this case-- is equivalent to a convolution in frequency-domain. Along the same lines, we can see that k^{th} power of $x(t)$ results in (k-1) convolutions in the frequency-domain (*auto-convolutions*):

$$x^k(t) \Leftrightarrow (\frac{1}{2\pi})^{k-1}.X(w) * X(w) * \cdots * X(w) \quad (3.25)$$

These repeated convolutions yield the output signal in the frequency-domain:

$$Y(w) = 2\pi a_0.\delta(w) + \sum_k a_k.[(1/2\pi)^{k-1}.X(w) * X(w) * \cdots * X(w)] \quad (3.26)$$

Computation of (3.26) is a major challenge unless $X(w)$ is a very simple function. For most real-life signals, we resort to numerical techniques to evaluate such a demanding task. Nevertheless, as a result of these auto-convolutions, we get signal distortion and interference to/from neighboring channels and signals due to pulse spreading. This is a particularly severe problem in frequency-division multiplexing (FDM) systems but a not major issue in TDM systems for the present day digital communication systems.

A quantitative measure of non-linear distortion is the second, third, fifth (ant other) harmonic distortion, which are found by taking a simple input: $x(t) = Cosw_0 t$ and substituting in 3.24:

$$y(t) = (\frac{a_2}{2} + \frac{3a_4}{8} + \cdots) + (a_1 + \frac{3a_3}{4} + \cdots)Cosw_0 t + (\frac{a_2}{2} + \frac{a_4}{4} + \cdots)Cos2w_0 t + \cdots \quad (3.27)$$

Then the second harmonic distortion is the ration of the second term in (3.27) to the fundamental harmonic:

$$2^{nd} \text{ Harmonic Distortion} = \left| \frac{\frac{a_2}{2} + \frac{a_4}{4} + \cdots}{a_1 + \frac{3a_3}{4} + \cdots} \right| x100\% \quad (3.28)$$

which is normally removed by filtering.

Example 3.3: Suppose that the input $x(t)$ and the output $y(t)$ of a channel are related through a quadratic equation defined by:

$$y(t) = x(t) + 0.001x^2(t) \quad (3.29)$$

and the input is a sampling sinc signal:

$$x(t) = \frac{1000}{\pi} Sinc(1000t) \quad (3.30)$$

Let us study the output signal both in the time-domain and the frequency-domain using standard results from Fourier tables:

$$y(t) = \frac{1000}{\pi} Sinc(1000t) + \frac{1000}{\pi^2} Sinc^2(1000t) \quad (3.31a)$$

$$Y(w) = \Pi(\frac{w}{2000}) + 0.316\,\Delta(\frac{w}{4000}) \quad (3.31b)$$

where "Π" and "Δ" represent rectangular and triangular waveforms, respectively. The second term in (3.31b) is the undesired (distortion) term in the received signal. Due to squaring, the bandwidth of the received signal has increased to 2000 rads/s, twice that of $x(t)$.

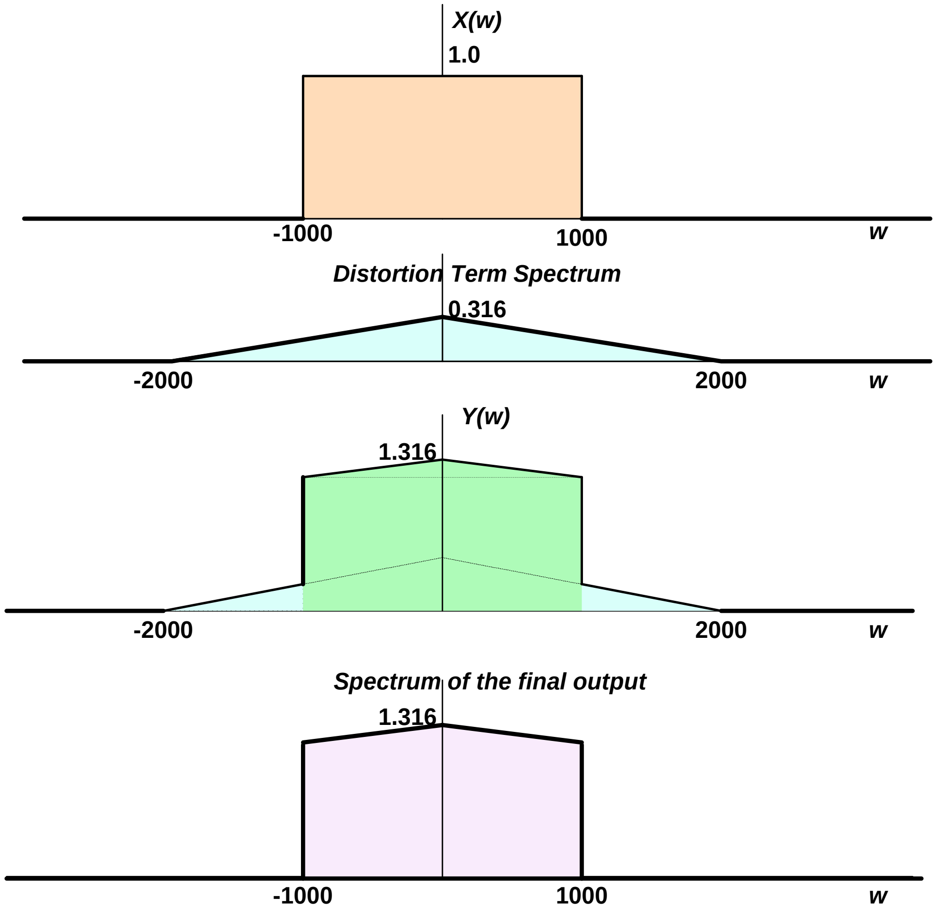


Figure 3.6 Spectra of the input, output and the distortion for Example 3.3.

As we can clearly observe from the last plot in Figure 3.6, the output from this channel is significantly different from the input. In particular, there is a triangular portion sitting on top of the original signal and hence, the inverse Fourier transform results in a distorted signal.

Multipath Distortion:

As the transmitted wave propagates through a given communication channel, such as the microwave transmission in the atmosphere, it is reflected from different layers of the medium due to non-homogenous character of the material. This results in splitting the waveform into multiple paths between the transmitter and the receiver. Some of these diverse waveforms will not reach the intended receiver's equipment at all. But some will do with attenuation and a time-delay due to variations in the path length.

Example 3.4: Let us consider the special case of two paths as shown in the diagram, where the top path is the direct line of sight between the source and the destination and assume it has negligible delay -i.e., zero in the figure below, and the lower path represents a reflected wave from ionosphere with a delay of τ seconds and an attenuation of α percent.

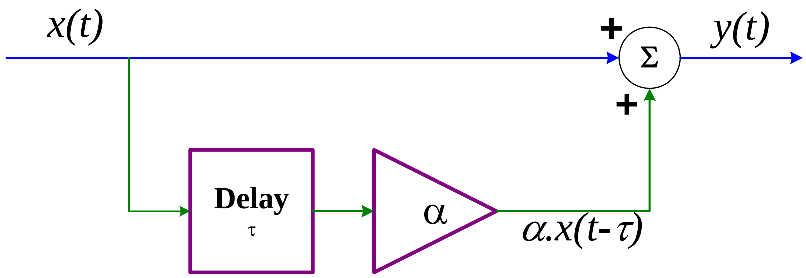


Figure 3.7 A simple multipath communication system model.

The total output is simply the input with no delay and its attenuated and delayed version:

$$y(t) = x(t) + \alpha.x(t-\tau) \tag{3.32a}$$

Let us take the Fourier transform of term-by-term and use the delay property to get:

$$Y(w) = X(w) + \alpha.X(w).e^{-jw\tau} \tag{3.32b}$$

The frequency response for this block diagram given by:

$$H(w) \equiv \frac{Y(w)}{X(w)} = \frac{X(w)+\alpha.X(w).e^{-jw\tau}}{X(w)} = 1+\alpha.e^{-jw\tau} = 1+\alpha.Cosw\tau - j\alpha.Sinw\tau$$

with magnitude and phase terms:

$$\begin{aligned} |H(w)| &= \sqrt{(1+\alpha.Cosw\tau)^2 + \alpha^2.Sin^2 w\tau} \\ &= \sqrt{1+2\alpha.Cosw\tau + \alpha^2.Cos^2 w\tau + \alpha^2.Sin^2 w\tau} \\ &= \sqrt{1+2\alpha.Cosw\tau+\alpha^2} \end{aligned} \tag{3.33a}$$

$$\theta_h(w) = -\arctan\left(\frac{\alpha.Sinw\tau t}{1+\alpha.Cosw\tau}\right) \tag{3.33b}$$

It is worth noting that the numerator and denominator terms are both periodic in w with a period $2\pi/\tau$, causing both magnitude distortion and phase distortion. This distortion can be partly corrected by a compensator system called "delay-line equalizer."

In addition to distortion, we have "fading" due to time-depending changes in the channel due to the compositional changers in the medium, including ambient temperature variations, day-to-night density changes, sunny to cloudy, humid to dry air, and many other environmental conditions. The fading is very pronounced in "cellular channels." One of the most common methods to tackle this problem is to use an Automatic Gain Control (AGC) mechanism. However, the success is only partial.

3.4 Transmission Loss

Power gain is the ratio of output average power to the input power:

$$g = \frac{P_{out}}{P_{in}} \tag{3.34a}$$

but almost all the time we use the decibel scale:

$$g_{dB} = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right) \tag{3.34b}$$

In many applications, power (both input and output) normalized to 1.0 Watt or 1.0 mW are also used:

$$P_{dBW} = 10\log_{10}\frac{P}{1W} \quad \text{and} \quad P_{dBm} = 10\log_{10}\frac{P}{1mW} \tag{3.35}$$

$$P^{out}{}_{dBm} = g_{dB} + P^{in}{}_{dBm} \quad (3.36)$$

For systems we also use the term relative gain:

$$|H(f)|_{dB} = 10\log_{10}(H(f)|^2 \quad (3.37)$$

Transmission Loss:

$$L = \frac{1}{g} = \frac{P_{in}}{P_{out}} \text{ and } L_{dB} = -g_{dB} = 10\log_{10}\frac{P_{in}}{P_{out}} \quad (3.38)$$

In the case of transmission lines, coaxial and fiber optic cables as the communication channel, the output power is known to attenuate (decrease) exponentially with distance and we write the power expression as:

$$P_{out} = 10^{-(\alpha l/10)}.P_{in} \quad (3.39)$$

where l is the path length between source and destination and α is the attenuation coefficient in dB per unit length. Then the loss can be expressed as:

$$L_{dB} = \alpha l \quad (3.40)$$

Table below shows typical values of transmission loss.

Table 3.3-1 Typical values of transmission loss

Transmission Medium	Frequency	Loss dB/km
Open-wire pair (0.3 cm diameter)	1 kHz	0.05
Twisted-wire pair (16 gauge)	10 kHz	2
	100 kHz	3
	300 kHz	6
Coaxial cable (1 cm diameter)	100 kHz	1
	1 MHz	2
	3 MHz	4
Coaxial cable (15 cm diameter)	100 MHz	1.5
Rectangular waveguide (5 × 2.5 cm)	10 GHz	5
Helical waveguide (5 cm diameter)	100 GHz	1.5
Fiber-optic cable	3.6×10^{14} Hz	2.5
	2.4×10^{14} Hz	0.5
	1.8×10^{14} Hz	0.2

Repeaters: Large attenuation necessitates re-amplification of the signal in the channel by systems called repeaters as discussed in p. 102 in Carlson.

Example 3.5: Satellite Relay System.
As shown in Figure 3.8 a transoceanic TV broadcast requires a geostationary satellite to be used as a repeater (relay). Consider the satellite at height 22,300 (36,000 km) miles above the equator with an uplink frequency of 6.0 GHz and a downlink one at 4.0 GHz. It is shown in Carlson that the free-space loss is given by:

$$L = \left(\frac{4\pi l}{\lambda}\right)^2 = \left(\frac{4\pi f l}{c}\right)^2 \quad (3.41)$$

Here λ is the wavelength. If we express f frequency in GHz, c speed of light and l distance in km we get:

$$L_{dB} = 92.4 + 20\log_{10} f_{GHz} + 20\log_{10} l_{km} \quad (3.42)$$

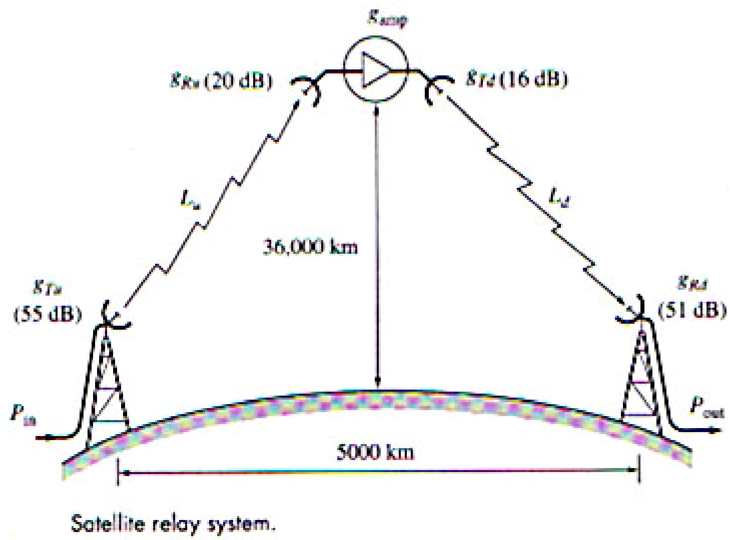


Figure 3.8 Satellite relay system performance

In this example, we have uplink and downlink losses as:

$$L_u = 92.4 + 20\log_{10} 6 + 20\log_{10} 36000 = 199.1\, dB$$

$$L_d = 92.4 + 20\log_{10} 4 + 20\log_{10} 36000 = 195.6\, dB$$

The satellite has a repeater amplifier with a typical output 18 dBW and uplink and downlink antenna gains of 20 and 16 dB, respectively.

If the transmitter outputs 35 dBW the satellite sees: 35+55-199.1+20=-144.1 dBW.

The power output at the receiver will be: 18+16-195.6+51=-110.6 dBW, which is typical for commercial satellites.

3.5 Introduction to Analog and Digital Filters

In almost every step of the information transmission process we are faced with shaping the spectrum of signals using various types of filters. More critically, the signal is shaped by the characteristics of the medium that is used as the channel. In communications systems community, the channel characteristics are presented in the frequency-domain and they behave like filters in many situations. Here, we will present the notion of filters with the general objective of eliminating, or suppressing unwanted components from information-bearing signal. Even though, the filters used in actual implementations have characteristics significantly different from their ideal counterparts, the latter ones are useful tools to develop the notion and the necessary terminology.

At first, analog filters will be introduced due to their simple description in the frequency-domain. However, with the ever increasing dominance of digital telecommunication systems and many related applications, analog filters are replaced by their digital counterparts. Exception to this will be the following two classes of filters:

(1) "anti-aliasing filters" used prior to the sampling process and

(2) the final "integrating filters" at the receiver output.

Digital Filters: As we have discussed in the case of continuous versus digital communication systems, there are numerous advantages in implementing filters digitally. Some of these features are:

1. Reuse of systems and devices for other tasks even within a particular application.
2. Lower cost (most significant factor for industry).

3. The filter precision is determined by the digital word-length rather than the discrete or lumped parameters used in the design.
4. Usage of simple elements, such as adders, multipliers, shift and delay operators.
5. Component accuracy and tuning do not pose serious concerns.
6. Higher-order filters are easily implemented from the lesser-order ones.
7. They can be modified easily by changing the specific algorithm employed, thereby resulting in shortened design turnover time and the cost.

Analog Ideal Filters:

Ideal filters exhibit distortionless characteristics over one or more bands and zero response elsewhere. For instance, the amplitude and phase characteristics of an ideal bandpass filter, which permits information in a specified band to pass through unaffected, are shown in Figure 3.9.

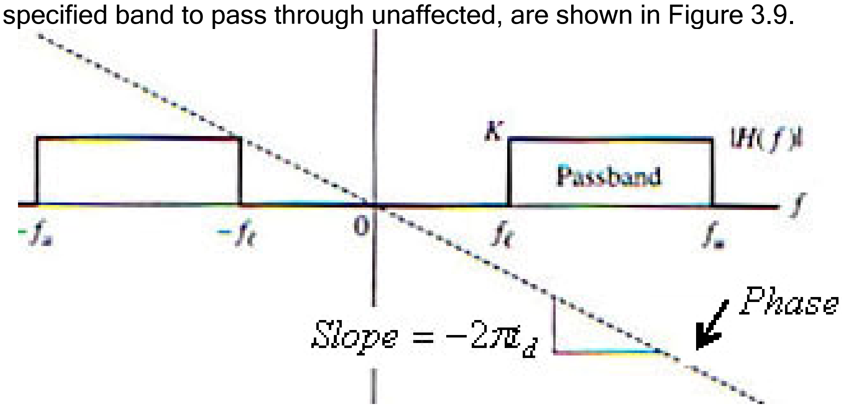


Figure 3.9 Amplitude and phase responses of an analog ideal bandpass filter.

The transfer function is given by:

$$H(f) = \begin{cases} K.e^{-j2\pi f.t_d} & f_l \le |f| \le f_u \\ 0 & Otherwise \end{cases} \quad (3.43)$$

where $\{f_l, f_u\}$ are the lower and upper edges of the passband and the phase response (argument) has a $Slope = -2\pi f.t_d$. The bandwidth of this filter is simply $B = f_u - f_l$ Hz.
The impulse response of this filter is found by taking inverse Fourier transform of (3.43) to yield:

$$h(t) = F^{-1}[H(f)] = 2BK.Sinc[B(t - t_d)].Cosw_c(t - t_d) \quad (3.44)$$

Similarly, ideal low pass filter is a special case of $f_l = 0 \quad with\ B = f_u$ Hz and Highpass similarly is another special case: $f_l > 0;\ f_u = \infty\ and\ B = f_l$.

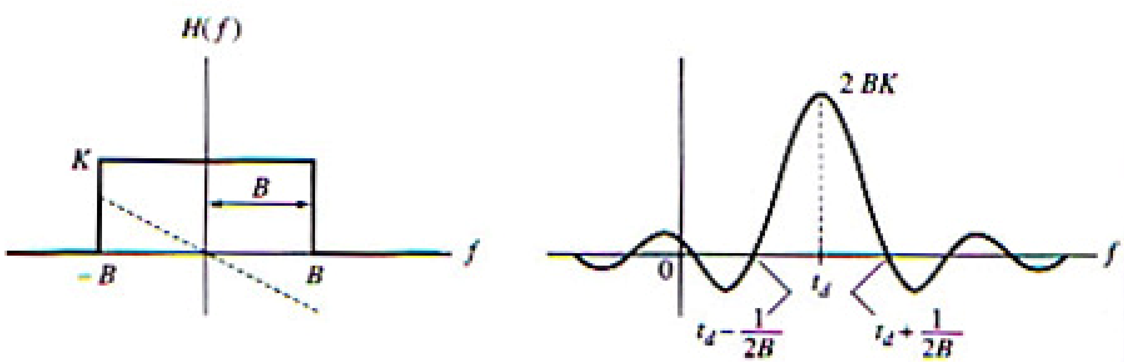


Figure 3.10 Frequency response and the impulse response of an ideal lowpass filter.

It is worth noting that the output signal is just the delayed version of the input by t_d seconds; i.e., it is transmitted without distortion:

$$y(t) = x(t - t_d) \quad (3.45)$$

It is easy to see that the frequency response is given by:

$$H(f) = K.\Pi(\frac{f}{2B}).e^{-jw.t_d} \quad (3.46)$$

and the impulse response for this filter can be readily obtained from the Fourier Tables:

$$h(t) = F^{-1}[H(f)] = 2BK.Sinc[2B(t - t_d)] \quad (3.47)$$

It is clearly observable from the impulse response of this filter as shown in Figure 3.10 that an anticipatory (non-causal or predictive, or forecasting) behavior is present. In other words, at a given time t_d we need to know the future values of the signal before it occurs. Therefore, this filter is not physically realizable.

Analog Real Filters:

In real-life, we use filters which do not exhibit perfect rectangular frequency responses. These filters have roll-off rates of varying degree depending upon the order of the filter at hand.

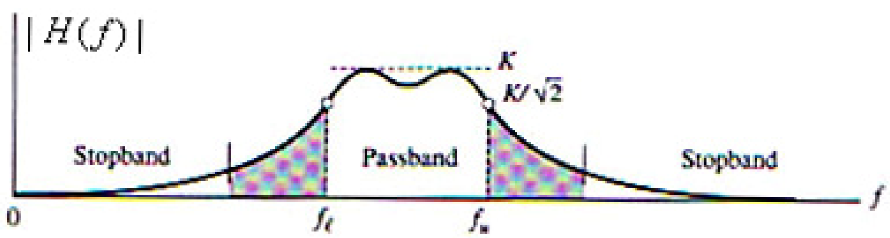


Figure 3.11 Amplitude response of a typical bandpass filter.

Passband segment of these filters are relatively large but not constant. The end points of the passband are defined as the 3-db bandwidth, or half-power bandwidth:

$$|H(f)| = \frac{1}{\sqrt{2}}|H(f)|_{\max} = \frac{K}{\sqrt{2}} = 0.707.K \quad when\ f_c \le f \le f_u \quad (3.48)$$

In other words, the power does not fall lower than $K^2/2$, hence the term half-power. In dB scale this corresponds to -3.0 dB and therefore, it is called 3dB bandwidth as well. Between the passband the the stopbands there are transition bands, which depend on the complexity of the filter under study.

Class of filters known as Butterworth filters is very frequently used as a yardstick because of their simplicity and implementation ease. These are realizable filters have no passband (in-band) ripple and also known as maximally flat. Transfer functions of Butterworth filters are also represented by a family of Butterworth polynomials P_n.:

$$H(f) = 1/P_n(j\frac{f}{B}) \text{ with property: } |P_n(j\frac{f}{B})|^2 = 1 + (\frac{f}{B})^{2n} \quad (3.49)$$

Passband frequency response is a non-increasing monotonic function of frequency. The magnitude of the frequency response of these filters are shown in Figure 3.12 and given by the expression:

$$|H(f)| = \frac{1}{\sqrt{1 + (\frac{f}{B})^{2n}}} \quad (3.50)$$

where n is the order of filter. Table 3.4-1 in Carlson lists the first four Butterworth polynomials.

Table 3.4-1 Butterworth polynomials

n	$P_n(p)$
1	$1 + p$
2	$1 + \sqrt{2}p + p^2$
3	$(1 + p)(1 + p + p^2)$
4	$(1 + 0.765p + p^2)(1 + 1.848p + p^2)$

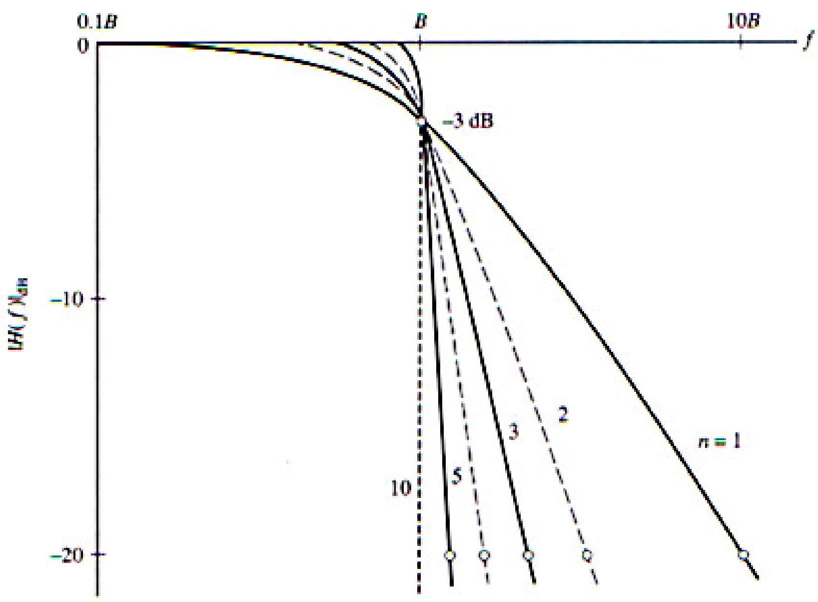


Figure 3.12 Amplitude Characteristics of Buttherworth filters of order $n = 1, 2, \cdots, 10$.

As explicit from (3.49) and the plots in Figure 3.12, Butterworth filters are completely specified by two parameters:

- The half-gain (3.0 dB) bandwidth B.
- The filter order N or n as used in (3.49) and (3.50).

Rule of Thumb:

- Amplitude of the frequency response of all Butterworth filters fall to 0.707 from their peak value of 1.0, also know as the **half-power rule**, -3dB point) at normalized frequency of $\frac{w}{2\pi B} = 1.0$
- Furthermore, the rate of roll-down is 6.0*N dB per octave, where N is the filter order.

Example 3.6: Analysis of a second-order Butterworth filter with $B = 1/2\pi.\sqrt{LC}$ will have a 12.0 dB roll-down every time the normalized frequency is doubled. In addition, the phase responses of these filters do not deviate significantly from the ideal linear phase characteristics until $f / B \geq 2.0$. This is sufficient for most practical communication systems. On the other hand, this behavior is not observed in many other filter classes.

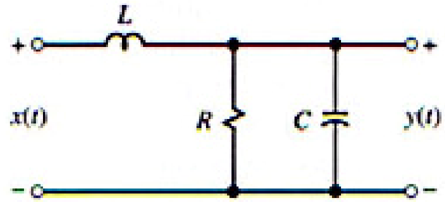


Figure 3.13 RLC filter as a 2nd order Butterworh filter.

The transfer function of this circuit can easily be obtained from Voltage Division Law in circuit theory:

$$H(s) = \frac{Z_{RC}}{Z_{RC} + sL} \Rightarrow H(f) = \frac{Z_{RC}}{Z_{RC} + jwL} \qquad Z_{RC} = \frac{R}{1 + jwRC}$$

$$H(f) = \frac{1}{1 + 1/jwL/R - w^2 LC} = [1 + j\frac{2\pi fL}{R} - (2\pi\sqrt{LC}\, f^2]^{-1}$$

Using the case for order two (2) from the table in Figure 3.12 and using $p = jf/B$, we have:

$$H(f) = [1 + j\sqrt{2}\frac{f}{B} - (\frac{f}{B})^2]^{-1}$$

These last two expressions gives us:

$\frac{2\pi L}{R} = \frac{\sqrt{2}}{B} = \sqrt{2}.2\pi.\sqrt{LC}$ and resulting at $R = \sqrt{\frac{L}{2C}}$

with this selection the above RLC circuit behaves just like a 2nd order Butterworth filter.

Pulse and Step Responses of Low Pass Filters:

The step and pulse responses of an ideal first-order are shown in Figure 3.14.

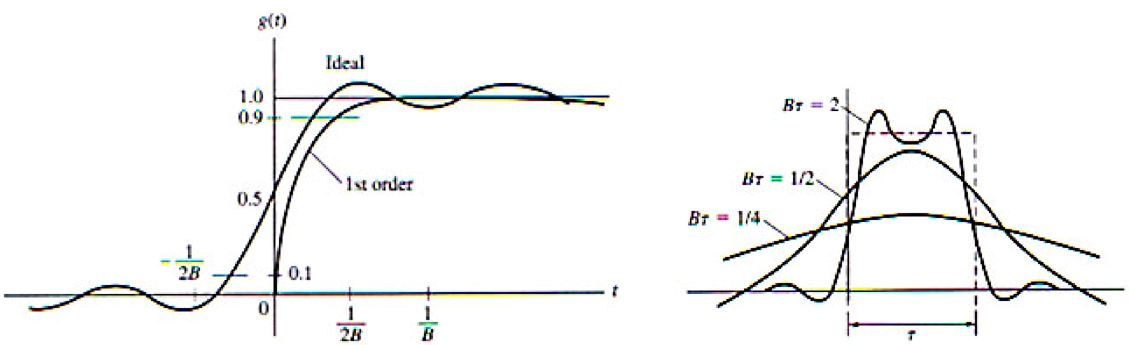


Figure 3.14 Step and Pulse responses of an ideal first-order LP filter.

Rise time t_r is defined as the time interval between 10%-90% of the rise of the signal. For instance, a careful measurement in the above curve yields $t_r \approx 0.35/B$. However, engineers use $t_r \approx 1/2B$ as a practical value in their calculations.

3.6 Correlation and Power Spectral Density (PSD)

If $v(t)$ is a power signal, the average power is defined in a generic fashion as the scalar (dot) product:

$$P_v = <|v(t)|^2> = <v(t).v^*(t)> \geq 0 \qquad (3.51)$$

where $< \bullet >$ stands for time-averaging:

$$< z(t) > = Lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} z(t).dt$$

with properties:

$$< z^*(t) >=< z(t) >^* ; < z(t-t_d) >=< z(t) >; < az_1(t) + bz_2(t) >= a < z_1(t) > +b < z_2(t) >$$

Schwarz's inequality:

$$|< v(t).v_w^*(t) >|^2 \le P_v.P_w \qquad (3.52)$$

Crosscorrelation:

$$R_{vw}(\tau) \equiv< v(t).w^*(t-\tau) >=< v(t+\tau).w^*(t) > \qquad (3.53)$$

where the independent variable τ is the time-difference between two signals. Crosscorrelation function satisfies:

$$| R_{vw}(\tau) |^2 \le P_v.P_w \qquad R_{vw}(\tau) = R_{vw}^*(-\tau)$$

Two signals $v(t)$ and $w(t)$ are uncorrelated if

$$R_{vw}(\tau) = 0 \quad \text{for all values of } \tau. \qquad (3.54)$$

If we correlate a signal with itself, we have autocorrelation function, which tells us about the time-variation of $v(t)$:

$$R_v(\tau) \equiv< v(t).v^*(t-\tau) >=< v(t+\tau).v^*(t) > \qquad (3.55)$$

with properties:

$$R_v(0) = P_v; \quad R_v(\tau) \le R_v(0); \quad R_v(-\tau) = R_v^*(\tau) \qquad (3.56)$$

Example 3.7: Autocorrelation of a sinusoid

Consider sum or difference of two signals $v(t)$ and $w(t)$:

$$z(t) = v(t) \pm w(t)$$

then

$$R_z(\tau) = R_v(\tau) + R_w(\tau) \pm [R_{vw}(\tau) + R_{wv}(\tau)] \qquad (3.57)$$

If these two signals are uncorrelated then

$$R_{vw}(\tau) = R_{wv}(\tau) = 0$$

and

$$P_z = P_v + P_w$$

Using these, it is easy to derive the autocorrelation of a sinusoidal signal: $z(t) = A.Cos(w_c t + \phi)$:

$$R_z(\tau) = \frac{A^2}{2}.Cos w_c \tau \qquad (3.58)$$

Correlation of Energy Signals:

Since averaging energy signals over all time yields zero, we can then talk about their total energy:

$$E_v = \int_{-\infty}^{\infty} v(t).v^*(t).dt \ \ge \ 0 \qquad (3.59)$$

The correlation functions can be similarly defined:

$$R_{vw}(\tau) = \int_{-\infty}^{\infty} v(t).w^*(t).dt \quad \text{and} \quad R_v(\tau) = \int_{-\infty}^{\infty} v(t).v^*(t).dt \tag{3.60}$$

We have the energy property:

$$|R_{vw}(\tau)|^2 \le E_v.E_w \quad \text{and}$$

the Parseval's energy theorem states that:

$$R_v(0) = E_v = \int_{-\infty}^{\infty} |V(f)|^2 .df = \int_{-\infty}^{\infty} v(t).v^*(t).dt = \int_{-\infty}^{\infty} |v(t)|^2 .dt \tag{3.61}$$

Schwarz's inequality:

$$\left| \int_{-\infty}^{\infty} V(f).W^*(f).df \right|^2 \le \int_{-\infty}^{\infty} |V(f)|^2 \, df . \int_{-\infty}^{\infty} |W(f)|^2 \, df. \tag{3.62}$$

Power (Energy) Spectral Density:

Fourier transform of autocorrelation function is defined as the power (energy) spectral density (PSD), which represents the distribution of power (energy) in the frequency-domain:

$$G_v(f) = F[R_v(\tau)] = \int_{-\infty}^{\infty} R_v(\tau).e^{-j2\pi f\tau}.d\tau \tag{3.63}$$

$$R_v(\tau) = G_v(f) = F^{-1}[G_v(f)] = \int_{-\infty}^{\infty} G_v(f).e^{+j2\pi f\tau}.df \tag{3.64}$$

Using the result in (3.58) we can compute the power spectral density (PSD) function of a sinusoidal signal $z(t) = A.Cos(w_c t + \phi)$ through Fourier transform:

$$G_z(f) = F[R_z(\tau)] = \int_{-\infty}^{\infty} \frac{A^2}{2}.Cosw_c\tau.e^{-j2\pi f\tau}.df = \frac{A^2}{2}.\int_{-\infty}^{\infty} .Cosw_c\tau.e^{-j2\pi f\tau}.df$$
$$= \frac{A^2}{4}\delta(f - f_0) + \frac{A^2}{4}\delta(f + f_0) \tag{3.65}$$

The two delta functions are displayed in Figure 3.15.

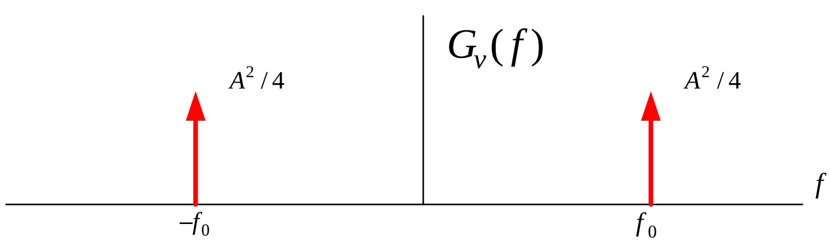


Figure 3.15 Power Spectral Density Function of a Sinusoid.

Autocorrelation Function for Passband (Modulated) Signals:

Let us modulate a generic time signal $x(t)$ with a sinusoid whose carrier frequency satisfies the Nyquist criterion, i.e., $w_0 \ge 2\pi B$, where B is the largest non-zero frequency component in $x(t)$:

$$z(t) = x(t).Cos(w_0 t) \tag{3.66}$$

From the modulation property of Fourier transforms of Chapter 2, we observe that the power spectral density (*PSD*) for the modulated signal will be equal to mirror-image frequency shifted versions of its baseband case, i.e.:

$$G_Z(w) = \frac{1}{4}[G_X(w + w_0) + G_X(w - w_0)] \quad (3.67)$$

and the autocorrelation function is simply the inverse Fourier transform of this result:

$$R_Z(\tau) = \frac{1}{2}.R_X(\tau)Cosw_0\tau \quad (3.68)$$

We observe that the modulation operation shifts the PDS of $x(t)$ by $\pm w_0$ and the power in the output signal is only half the power of $x(t)$:

$$P_Z = 1/2.P_X \quad for\ w_0 \geq 2\pi B \quad (3.69)$$

Suppose that one such signal is transmitted over a channel with a frequency response $H(w)$ and the output signal is $y(t)$. The frequently asked task is to compute the PSD and autocorrelation function of the output in terms of that of the input signal. The output PSD is given by

$$S_y(w) = |H(w)|^2.S_x(w) \quad (3.70)$$

and the corresponding autocorrelation function is obtained from the inverse Fourier transform:

$$R_y(\tau) = F^{-1}\{S_y(w)\} = h(\tau) * h(-\tau) * R_S(\tau) \quad (3.71)$$

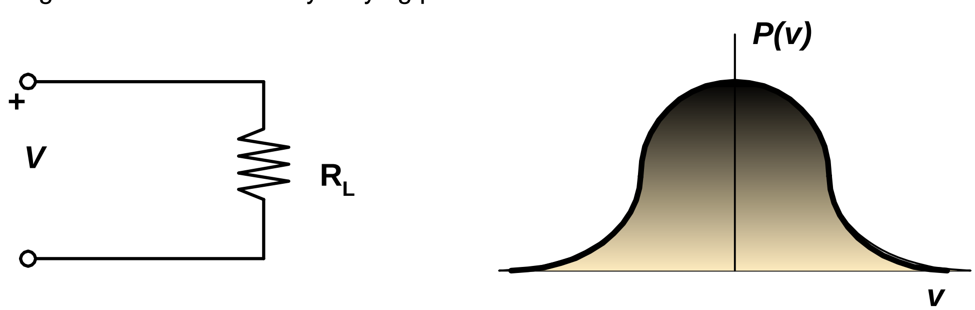
By substituting the PSD and autocorrelation functions of the input, the output characteristics are usually obtained from (3.70) and the inverse FT (middle equality in (3.71) not from the double convolution of the last term.

3.7 Noise in Communication Signals

Noise is defined as any unwanted energy that accompanies an information-bearing signal in a communication system. A signal at any point along its path from source to the user is almost always subject N noise which is in general due to the cumulative effects of similar and non-similar causes, which include interference from other sources and channels, thermal characteristics of components, atmospheric conditions, man-made noise, echoes, flicker noise, and quantizing noise emanating from representing continuous functions in terms of finite-length digital words. We will briefly discuss a few of these except the quantizing noise or distortion which will be discussed in the next chapter.

Thermal Noise:

Within a conductor, a resistor, or a transistor free electrons are produced because of thermal agitation. These electrons move "randomly" resulting in a rate of arrival at each end that also varies in a random manner. The randomness is best probabilistically represented in terms of a probability density function (PDF). The actual law governing the thermal noise is known as the Gaussian distribution. This random phenomenon gives rise to a randomly varying potential difference across the ends of a device.



a. Resistive Load b. PDF for the Thermal Voltage across Resistive Load

Figure 3.16 Thermal Noise Voltage across a resistive load and its Gaussian PDF.

Thermal noise usually has a mean (average) value of zero but a finite power. For instance, the noise voltage across a resistor R_L of Figure 3.16.a is given by

$$\sigma_V^2 = E\{V^2\} = \frac{2(\pi kT)^2}{3h} R_L \tag{3.72}$$

where:

k : Boltzman's constant = 1.38×10^{-23} *W/kHz* or Joule/degree Kelvin
T : Absolute Temperature in Kelvin degrees, K^o,
h : Planks constant = $6.62\text{x}10^{-34}$ Joules second
σ_V^2 : Variance of the Gaussian distribution

Furthermore, it can be shown that the power spectral density of the thermal noise is:

$$S_V(f) = \frac{2hR_L|f|}{e^{\frac{h|f|}{kT}} - 1} \tag{3.73}$$

A plot of $S_V(f)$ versus f is shown in Figure 3.17.

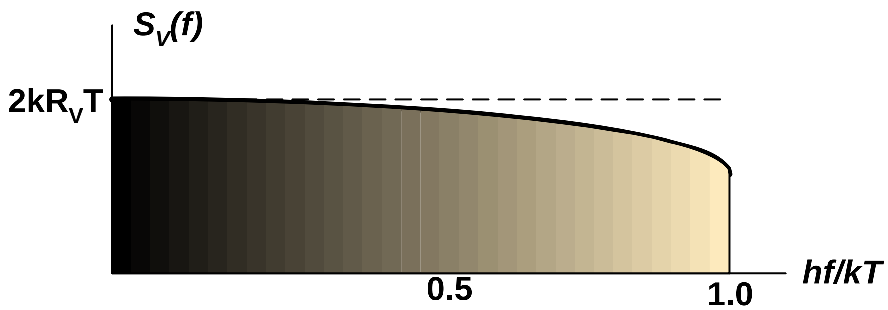


Figure 3.17 Power Spectral Density of Thermal Noise versus normalized frequency.

We see that the plot of PSD is fairly flat over the frequency range $-0.1\frac{kT}{h} < f < 0.1\frac{kT}{h}$. At room temperature it becomes $-10^{12} < f < 10^{12}$ *Hz*. This frequency range is well outside the range of frequency bands used in conventional communication systems. Hence, for the purposes of modeling, the PSD of thermal noise in electrical systems can be assumed flat. The term commonly used for this characteristics is "*White Noise*."

What is important in practice is the noise delivered into a load. Usually, in a transmission system, the source resistor R_S and the load resistor R_L are matched to achieve maximum power transfer. For such a system, the maximum power delivered to the load R_L is equal to:

$$P_L = E\{V^2\}\frac{R_L}{(R_S + R_L)^2} = \frac{E\{V^2\}}{4R_L} \tag{3.74}$$

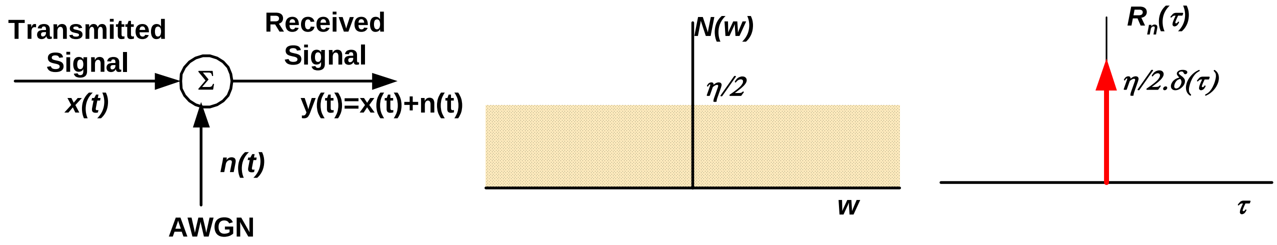
Extending this concept to the thermal resistor viewed as the noise source in the system, the available power spectral density at the matched load becomes:

$$S_V(f) = \frac{kT}{2} \ \ watts/Hz \tag{3.75}$$

In many applications, this will be the PSD of the white noise with a level $\eta/2$ as discussed below.

Additive White Gaussian Noise:

It is common practice in communication systems and information theory to use an additive zero-mean Gaussian Noise with a flat spectrum. This is known as the Additive White Gaussian Noise (AWGN) model for noisy communication regimes. The communication system block diagram for this case is shown in Figure 3.18, where the transmitted signal $x(t)$ is corrupted by an additive noise *n(t)*, which exhibit normal or Gaussian characteristics with zero mean.



(a) AWGN Communication Model (b) White Noise Spectrum (c) Autocorrelation function

Figure 3.18 Additive White Gaussian Noise System Model, spectrum, and autocorrelation.

The Gaussian (normal) probability density function with a zero mean can be expressed by:

$$P(n) = \frac{1}{\sqrt{2\pi N_0}} e^{-\frac{n^2}{2N_0}} \tag{3.77}$$

where N_0 is the total noise power over all frequencies, or the variance of the distribution. The power spectral density PSD of the above white noise is simply $\eta / 2$ from Figure 3.16.b. The inverse Fourier transform yields the autocorrelation function for this function:

$$R_n(\tau) = F^{-1}\{S_n(w)\} = \frac{\eta}{2}.\delta(\tau) \tag{3.78}$$

It is a single delta function at the origin. The total power in the signal can be found either by integrating PSD over all frequencies or simply from the result in (3.57), i.e.

$$P_T = R(0) \tag{3.79}$$

In Appendix 3.A, we have generated Gaussian random numbers of various sizes to demonstrate how large the size of the test samples should be in order to approximate a smooth Gaussian distribution.

If a white noise with a *PSD* $\eta / 2$ is transmitted through a linear system with a frequency response $H(w)$, it can be shown that the output signal from this process will have a power spectral density:

$$Y(w) = |H(w)|^2 .S_x(w) \tag{3.80}$$

and the overall power for this can be found by integrating $Y(w)$ for all frequencies, $-\infty < f < \infty$. The autocorrelation function, however, is found by taking inverse Fourier transform of $Y(w)$.

Low-pass Filtered White Gaussian Noise

A signal is low-pass filtered or band-limited white Gaussian noise if has Gaussian PDF (statistics) and the power spectral density has a finite bandwidth as shown in Figure 3.19a., i.e.,

$$S_X(w) = \begin{cases} \eta/2 & -w_B \le w \le w_B \\ 0 & Otherwise \end{cases} \tag{3.81}$$

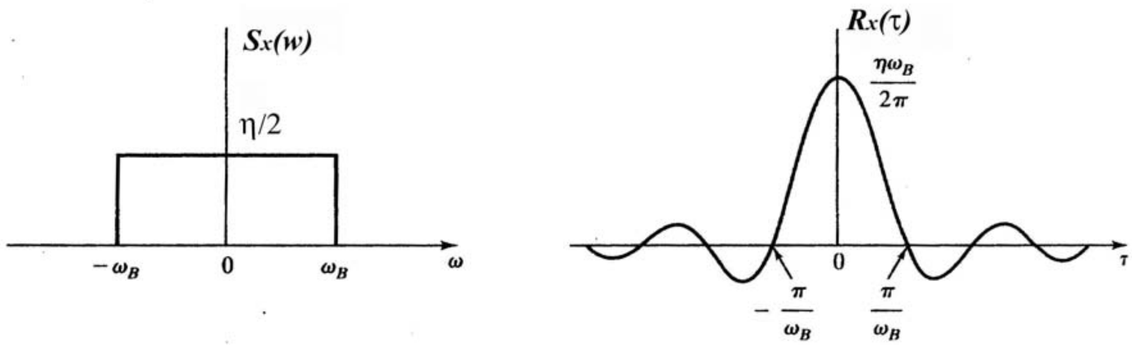


Figure 3.19 Power spectral density and autocorrelation function of and band-limited white noise.

The inverse Fourier transform of this gives us the autocorrelation function, which exhibits Sinc characteristics as depicted in Figure 3.19b.

$$
\begin{aligned}
R_X(\tau) &= \frac{1}{2\pi}\int_{-w_B}^{w_B} S_x(w)e^{jw\tau}dw = \frac{1}{2\pi}\int_{-w_B}^{w_B}(\eta/2).e^{jw\tau}dw \\
&= \frac{\eta w_B}{2\pi}.\frac{Sin(w_B\tau)}{w_B\tau}
\end{aligned} \qquad (3.82)
$$

It is worth noting that $R_X(\tau)$ in (3.82) is not time-limited. However, in the limit as $w_B \to \infty$ or equivalently, as $\tau \to 0$ the autocorrelation function is approaching a Dirac delta (impulse) function. In practice, band-limited white noise is also known as "pink noise."

Band-pass Filtered White Gaussian Noise

Let us now consider the zero-mean Gaussian noise with a spectral power density $\eta/2$ has an ideal band-pass characteristic with a bandwidth of $2W_B$ radians/second as shown in Figure 3.20. This is the most frequently observed characteristics in radio communication with a line of sight. In order to find the autocorrelation function for this noise we need perform the inverse Fourier transform over the non-zero intervals of the spectrum.

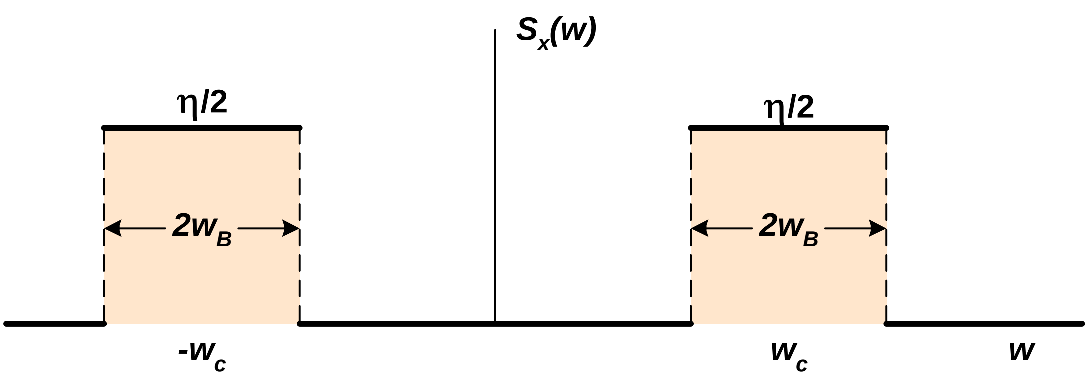


Figure 3.20 PSD of an ideal band-pass filtered white noise.

$$
\begin{aligned}
R_X(\tau) &= \int_{-w_c-w_B}^{-w_c+w_B}(\eta/2).e^{jw\tau}dw + \int_{w_c-w_B}^{w_c+w_B}(\eta/2).e^{jw\tau}dw \\
&= \frac{\eta}{2}.2w_B.Sinc(\frac{4w_B}{2\pi}\tau).[e^{-jw_c\tau} + e^{+jw_c\tau}] \\
&= \frac{2\eta w_B}{\pi}Sinc(\frac{w_B}{\pi}\tau).Cos(w_c\tau)
\end{aligned} \qquad (3.83)
$$

The plot of this result is shown in Figure 3.21.

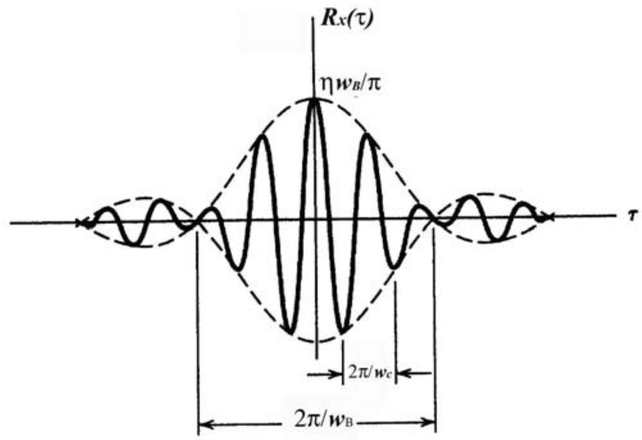


Figure 3.21 Autocorrelation Function of an ideal band-pass filtered white noise.

Narrow-Band White Noise

A front end of the receiver in many communication systems include a filter to suppress all frequency components outside a narrow band of interest for that communication link, such as the carrier frequency f_C allocated by FCC in the US for a radio station. In the absence of any information bearing signal, the output of this filter is a pure noise signal with spectral components centered in the neighborhood of that particular carrier frequency, commonly known as the "*narrow-band white noise*".

The term narrow comes from the fact that the neighborhood is very small in comparison to the complete spectrum available for communication. To study the effects of narrow-band noise on the performance of communication systems, we use one of the following two representations:

- Formulation in terms of in-phase and quadrature components.
- Formulation in terms of envelope and phase terms.

We will first present the in-phase and quadrature method using the setup in Figure 3.22 and in the following canonical form:

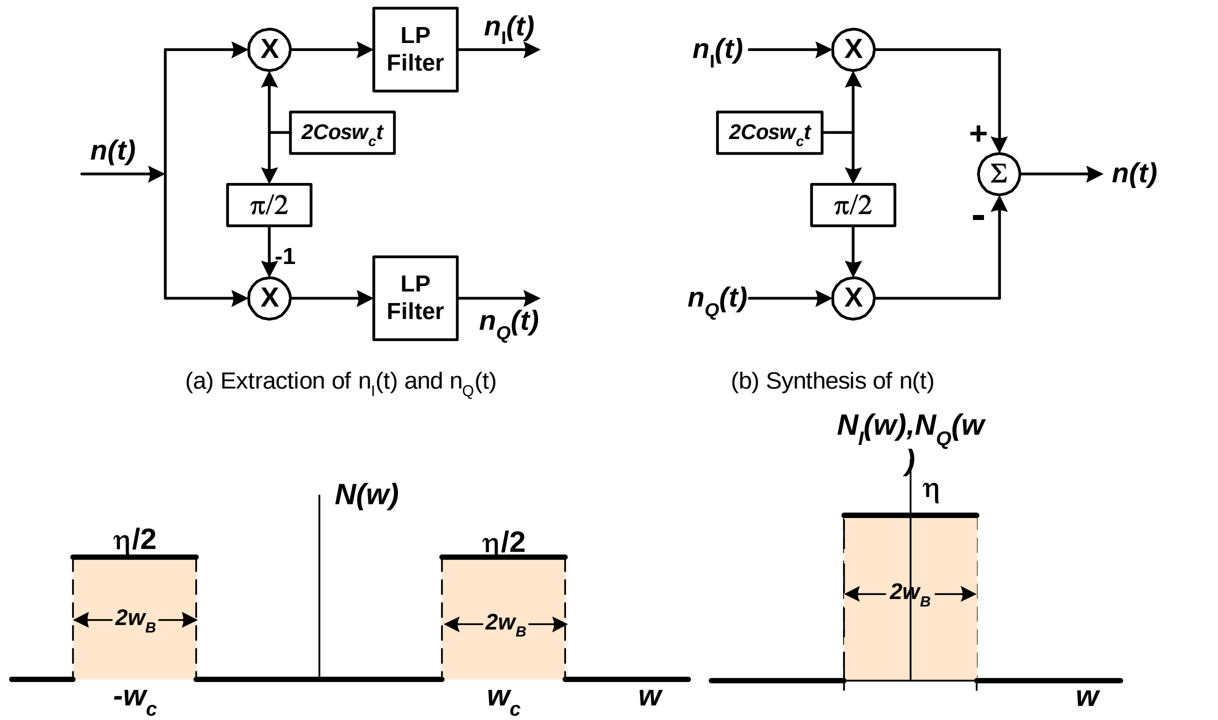
$$n(t) = n_I(t).Cos(w_c t) - n_Q(t).Sin(w_c t) \tag{3.84}$$

where:

$n_I(t)$: in-phase noise signal component and

$n_Q(t)$: Quadrature noise component.

Because of the cosine and sine functions associated with these components, they are also called cosine and sine term in the industrial community.



(a) Extraction of $n_I(t)$ and $n_Q(t)$ (b) Synthesis of n(t)

(c) Passband spectrum of Narrow band white noise (d) Spectra of In-phase and Quadrature terms

Figure 3.22 (a) Narrow-band (*NB*) noise analysis stage, (b) Synthesis in terms of $n_I(t)$ and $n_Q(t)$, (c.) Spectrum of *NB* white noise, and (d) Spectra of in-phase and quadrature terms.

Observations:

- In order to use the term narrow-band (NB) it is critical that the bandwidth of the signal is significantly smaller than that of the carrier frequency: $2W_B << W_c$. In practice, a few percent is normally used.
- Both $n(t)$ and its components $n_I(t)$ and $n_Q(t)$ have zero mean and finite variance, or equivalently, power.
- If $n(t)$ is Gaussian, then the in-phase and quadrature terms are also Gaussian.