Chapter 2: SIGNALS AND SPECTRA

In the previous chapter we have defined signals used in communication systems. We have classified them as analog or digital, baseband or bandpass, and deterministic or random, energy or power. We have further noted that random signals are important as far as transmission of information is concerned. In this chapter, we will develop mathematical descriptions for signals used in communication systems in the time-domain and introduce Fourier analysis in the frequency-domain.

2.1 Line Spectra and Fourier Series

2.1.1 Phasors and Line Spectra:

Consider a sinusoidal (a.c.) voltage $v(t)$:

$$v(t) = A.Cos(w_0 t + \phi) \tag{2.1}$$

where *A* is the amplitude (peak value, strength) and w_0 is the radian frequency as defined in the previous chapter. The phase angle ϕ represent the shift in $v(t)$ at the time origin: $t = 0$. It is clear from (2.1) that the voltage is periodic with a fundamental period: $T_0 = 2\pi / w_0$ and its reciprocal is called the fundamental frequency:

$$f_0 \equiv \frac{1}{T_0} = \frac{w_0}{2\pi} \text{ in Hertz (Hz).} \tag{2.2}$$

The signal in (2.1) is very frequently represented by a complex exponential or phasor as it is called by the circuits community and it is based on Euler's Theorem:

$$e^{\pm j\theta} = Cos\theta + j.Sin\theta \tag{2.3}$$

where $j \equiv \sqrt{-1}$ and the angle is represented by its real and imaginary terms: $\theta = w_0 t + \phi$. Then the sinusoidal signal of (2.1) can be with a phasor representation by:

$$v(t) = A.Cos(w_0 t + \phi) = A.\mathrm{Re}\{e^{j(wt+\phi)}\} = \mathrm{Re}\{Ae^{j\phi}.e^{jwt}\} \tag{2.4}$$

The first term of the product is known as the amplitude and the last one the phase of the signal. In Figure 2.1 we show the phasor-domain representation and the frequency-domain description. Since there is only one frequency value associated with this signal f_0 or w_0, there will be a straight line of height A at location f_0 Hz in the frequency plot. In other words, we have a "line spectrum."

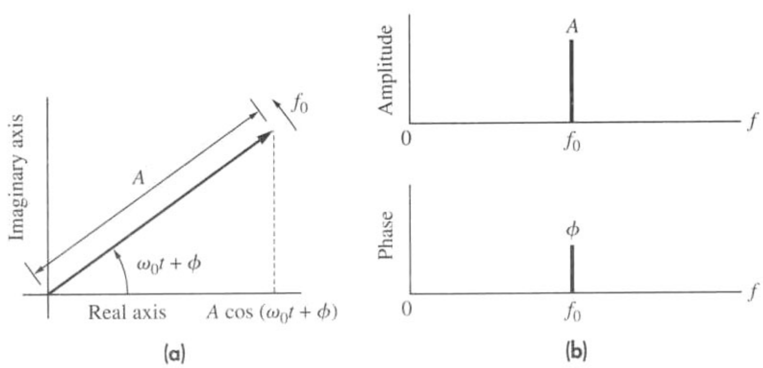


Figure 2.1 Phasor and line spectral representations of a sinusoidal voltage. (Carlson: Figure 2.1-2)

Amplitudes have always positive values and negative signs are absorbed in the phase using:

$$-A.Coswt = A.Cos(wt \pm \pi) \tag{2.5}$$

Example 2.1: Consider the following signal:

$$w(t) = 7 - 10Cos(40\pi t - \pi/3) + 4Sin(120\pi t)$$
$$= 7.Cos(2\pi.0.t) + 10Cos(20.2\pi t) + 4Cos(60.2\pi t - \pi/2)$$

First line of this equation is displayed in Figure 2.2. In the second line we have three terms oscillating with 0, 20, & 60 Hz, therefore, there will be 3 harmonics located at those line frequencies and three phase values in phase plots. It is worth noting that the frequency-domain plots, amplitude and phase, are one-sided or positive frequency line spectra.

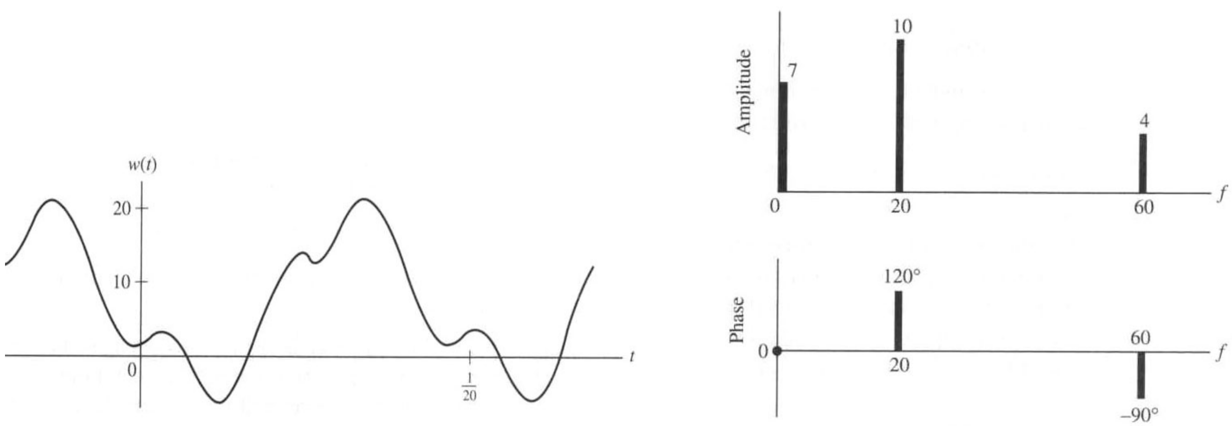


Figure 2.2 Time-domain, amplitude and phase plots of $w(t)$ in Example 2.1. (Carlson: 2.1-3)

However, we discuss Fourier series and Fourier transforms we will see that two-sided spectral representation incorporating negative frequency (mirror image) notion is more useful and plots from all analytical results and computer examples will be two-sided. The reason for this comes from Euler's result: $RE[z] = 1/2.(z + z^*)$ where z is any complex variable and z* is its complex conjugate. With this in mind, we can write (2.4) as shown in Figure 2.3:

$$\begin{aligned} v(t) &= A.Cos(w_0 t + \phi) = A.\mathrm{Re}\{e^{j(w_0 t + \phi)}\} = \mathrm{Re}\{Ae^{j\phi}.e^{jw_0 t}\} \\ &= A/2.e^{j\phi}.e^{jw_0 t} + A/2.e^{-j\phi}.e^{-jw_0 t} \end{aligned} \tag{2.6}$$

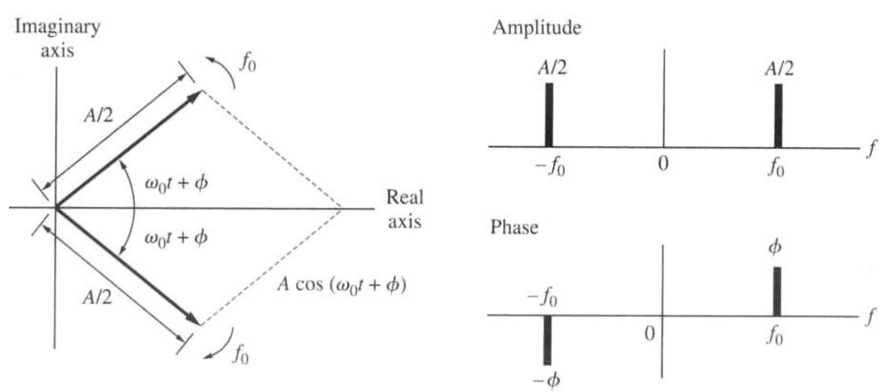


Figure 2.3. Phasor and Amplitude plots of (2.6) (Carlson: 2.1-4)

If we work example 2.1 with two sided spectra similar to (2.6) we will obtain slightly different amplitude and phase plots, each with mirror images and half strengths as shown in Figure 2.4.

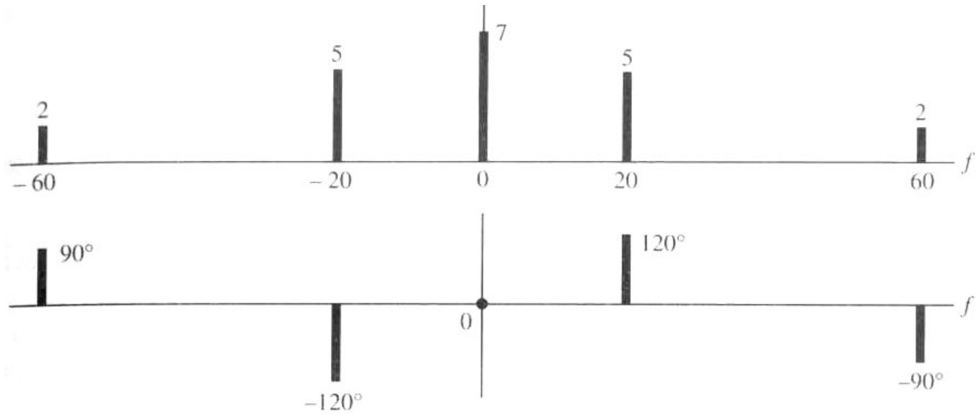


Figure 2.4 Two-sided amplitude and phase spectra of Example 2.1. (Carlson 2.1-5)

Periodic Signals and Average Power: A signal $v(t)$ is periodic if there is a positive constant T_0, such that

$$v(t) = v(t + mT_0) \qquad for \;\; all \;\; t \tag{2.7}$$

The smallest value of T_0 that satisfies the above periodicity condition is called the fundamental period of $v(t)$. A signal is aperiodic (non-periodic) if it does not satisfy (2.7).

Basic Properties:

1. A Periodic signal $v(t)$ has the same shape when it is observed in any other full period.
2. Periodic signals always start at $t = -\infty$ and continue all the way to $t = +\infty$.
3. Period signal with a period T_0 is also periodic with mT_0, where m is any positive integer.

Energy in a real signal $v(t)$ is defined by:

$$E = \int_{-\infty}^{\infty} v^2(t)dt \tag{2.8}$$

If $v(t)$ is a complex signal, as in some communication applications, the energy definition needs to be modified to:

$$E = \int_{-\infty}^{\infty} | v(t) |^2 dt \tag{2.9}$$

However, the energy concept will not be meaningful for deterministic or periodic signals and it will result in infinity. For almost all of our systems applications, we use power as the notion to make meaningful inference about signals, systems, and their performance.

Average Power (time-average of energy) in a real signal, if it exists, is defined by:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} v^2(t)dt \qquad \text{for real signals} \tag{2.10}$$

and

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |v(t)|^2 dt \qquad \text{for complex signals} \tag{2.11}$$

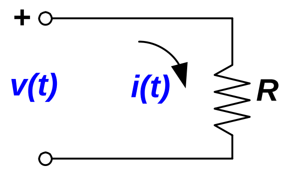
In signal processing terminology, P is the mean-squared value of $v(t)$ and finally, the square-root of P is the familiar root mean square (rms) value.

Important Observations:

1. Energy Signals: If $0 < E < \infty$; i.e., signal has a finite energy then $x(t)$ is called an energy signal. Note: All energy signals have ZERO power.
2. Power Signals: If $0 < P < \infty$; i.e., signal has a finite power then $x(t)$ is a power signal. Note: All power signals have INFINITE energy.

Example 2.2: Consider a simple resistive circuit.
Let us compute power and energy the load resistor.
The equation governing this resistive circuit from Ohm's Law:

$$v(t)=R.i(t)$$



The average power dissipated across this resistor is given by:

$$P = i^2(t).R = \frac{v^2(t)}{R} \quad Watts \tag{2.12}$$

and the energy dissipated across a load resistor of R Ohms is equal to:

$$E = \int_{-\infty}^{\infty} \frac{|v(t)|^2}{R} dt \quad Joules \tag{2.13}$$

Example 2.3.A: Consider a sinusoidal signal: $x(t) = A..Sin(w_0 t + \phi)$ with a period: $T = 2\pi / w_0$

Power over one full period T is given by

$$\begin{aligned} P_I &= \int_0^T (A^2/T).Sin^2(w_0 t + \phi)dt = (A^2/T).\int_0^T \left[\frac{1}{2} - \frac{1}{2}Cos(2w_0 t + 2\phi)\right]dt \\ &= (A^2/T).T/2 = A^2/2 \end{aligned} \tag{2.14}$$

Note that the second term in the integral above goes to zero. So, the power is finite. The energy in this signal would be the sum of the energy terms contributed from each period times the period T and there are infinitely many repetitions of this signal shape from minus infinity to plus infinity and the energy is:

$$E_{Total} = \frac{A^2}{2}T + \frac{A^2}{2}T + \cdots + \frac{A^2}{2}T \to \infty \tag{2.15}$$

Therefore, simple sinusoidal periodic signals have infinite energy and it is a power signal.

Example 2.3.B: Give an exponential signal $v(t) = A.e^{-t}$ measured in volts, find its energy and power content.

$$E = \int_{-\infty}^{\infty} \left|A.e^{-t}\right|^2 .dt = 2\int_0^{\infty} (A^2.e^{-2t})dt = \frac{2A^2}{-2} e^{-2t}\Bigg|_0^{\infty} = A^2 \quad \text{Joules} \tag{2.16}$$

which has a finite value; then $v(t)$ must be an energy signal and it must have $P = P_{av} = 0$.
Let us verify that:

$$\begin{aligned} P &= Lim_{L\to\infty}\left[\frac{1}{2L}.\int_{-L}^{+L}|v(t)|^2 dt\right] = Lim_{L\to\infty}\left[\frac{A^2}{2L}.\int_{-L}^{+L}| e^{-2t} | dt\right] = Lim_{L\to\infty}\left[\frac{2A^2}{2L}.\int_0^L e^{-2t}dt\right] \\ &= A^2 Lim_{L\to\infty}\left[\frac{1}{2L}.\frac{1}{-2}e^{-2t}\Big|_0^L\right] = \frac{A^2}{-4}Lim_{L\to\infty}\left[\frac{1}{L}.(e^{-2L} - 1)\right] = 0 \end{aligned} \tag{2.17}$$

The evaluation in the limit as $L \to \infty$ has resulted in zero power as expected.

2.1.2 Fourier Series:
Let $v(t)$ be a periodic power signal with a period: $T_0 = 1/f_0 = 2\pi / w_0$ then we can write its complex exponential (complex Fourier) series expansion from calculus as:

$$v(t) = \sum_{n=-\infty}^{\infty} c_n . e^{jnw_0 t} = \sum_{n=-\infty}^{\infty} c_n . e^{jn2\pi f_0 t} \tag{2.18}$$

which is known in the signals and systems community as the synthesis equation. The Fourier coefficients c_n are complex quantities and they can be obtained from another equation, commonly known as the analysis equation:

$$c_n = \frac{1}{T_0} \int_{T_0} v(t).e^{-jnw_0 t} dt = \frac{1}{T_0} \int_{T_0} v(t).e^{-jn2\pi f_0 t} dt \tag{2.19}$$
$$= | c_n | .e^{j\theta_n}$$

here the magnitude and phase (angle, argument) of these exponential Fourier coefficients are defined by:

$$| c_n | = \sqrt{\mathrm{Re}^2\{c_n\} + \mathrm{Im}^2\{c_n\}} = \sqrt{c_n . c_n^*} \quad \text{and} \quad \theta_n = \arg(c_n) = \arctan(\frac{\mathrm{Im}\{c_n\}}{\mathrm{Re}\{c_n\}}) \tag{2.20}$$

with these we can write (2.18) in terms of amplitudes and phase terms:

$$v(t) = \sum_{n=-\infty}^{\infty} c_n . e^{jn2\pi f_0 t} = \sum_{n=-\infty}^{\infty} | c_n | .e^{j\theta_n} .e^{jn2\pi f_0 t} \tag{2.21}$$

Observations:

1. $v(t)$ consists of phasors with amplitude $|c_n|$ and phase $\theta_n = \arg(c_n)$ located at frequencies: $nf_0 = 0, \pm f_0, \pm 2 f_0, \pm 3 f_0, \cdots$ This implies a two-sided line spectra in the frequency-domain and $| c(nf_0) |$ is the amplitude spectrum at those frequencies.

2. All frequencies are integer multiples (harmonics) of f_0. This line spectra is spaced uniformly.

3. The dc component is the time average of the signal:

   $$c_0 = c_{n=0} = c(0) = \frac{1}{T_0} \int_{T_0} v(t) dt = \langle v(t) \rangle$$

   where <*> stands for the time-averaging operation.

4. If $v(t)$ is real then $c_{-n} = c_n^* = | c_n | .e^{-j\theta_n}$

   with $| c_{-n} | = | c(-nf_0) | = | c_n | = | c(nf_0) |; \arg\{c(-nf_0)\} = -\arg\{c(nf_0)\}$ (2.22)

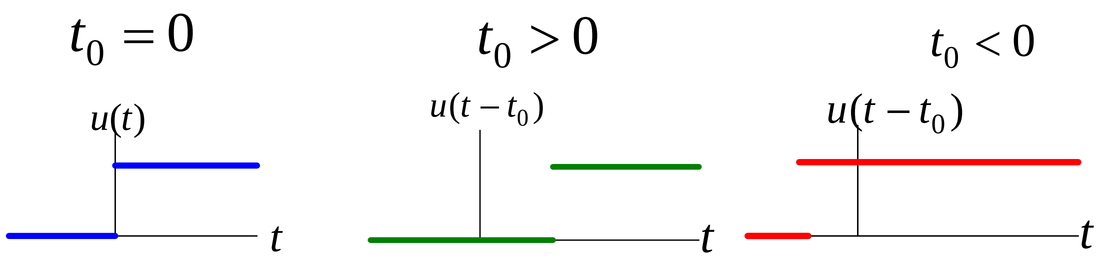
5. When dealing with complex signals, we combine complex conjugate pairs to generate a more compact form:

   $$v(t) = c_0 + \sum_{n=1}^{\infty} | 2c_n | .Cos(2\pi f_0 nt + \theta_n) \tag{2.23}$$

   which is known as the trigonometric (cosine) Fourier series.

2.1.3 Important Signals used in Communications:
Unit-step function $u(t)$ and the generic step function are defined by the following inequalities:

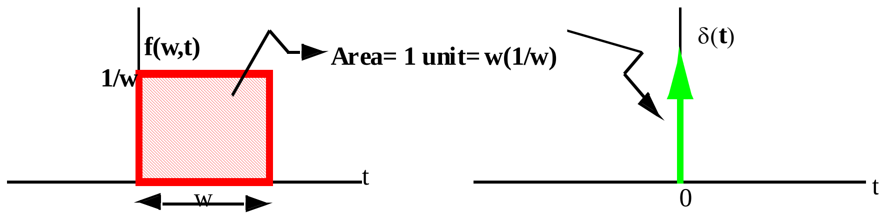
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \quad \text{and} \quad u(t - t_0) = \begin{cases} 1 & t > t_0 \\ 0 & t < t_0 \end{cases} \tag{2.24}$$



Note: *u(t)* is undefined at *t=0*; i.e., it is discontinuous at zero.

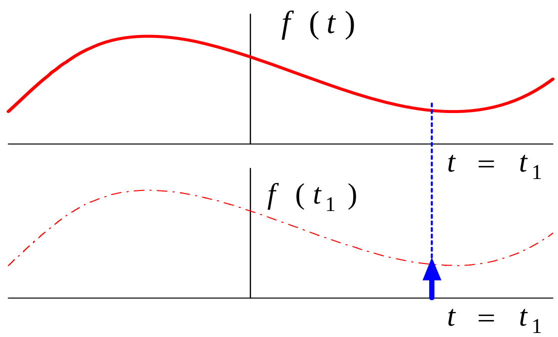
Unit-impulse (Dirac Delta) function: $\delta(t)$ is a limiting behavior of a narrow pulse at origin with an area equal to unity.

$$\delta(t) = \lim_{w \to 0} f(w,t) \quad = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases} \tag{2.25}$$



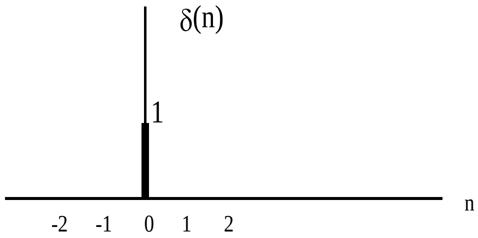
In other words, Dirac Delta function has a zero width and infinite height with unit area. This description of delta function is a conceptual definition. However, the formal definition of a unit impulse function is normally done through what is known as the Sifting Theorem in the field of applied mathematics or the sampling property in the signal processing community.
Sifting Theorem definition of a unit impulse (delta) function:

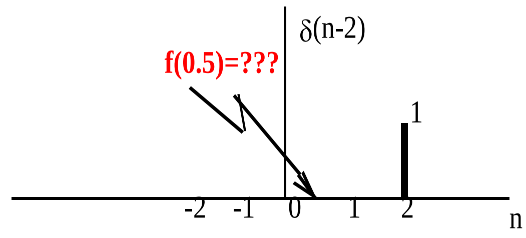
$$f(t_1) = \int_{-\infty}^{\infty} f(t).\delta(t - t_1)dt \tag{2.26}$$



Unit-sample sequence:

$$\delta[n] = \begin{cases} 1 & if \ n = 0 \\ 0 & Otherwise \end{cases} \quad \text{and} \quad \delta[n-k] = \begin{cases} 1 & if \ n = k \\ 0 & if \ n \neq k \end{cases} \tag{2.27}$$





Any discrete signal $f[n]$ can be represented by a sum of appropriately weighted unit-sample functions:

$$f[n] = \sum_{m=-\infty}^{\infty} f[m].\delta[n-m] \qquad (2.27)$$

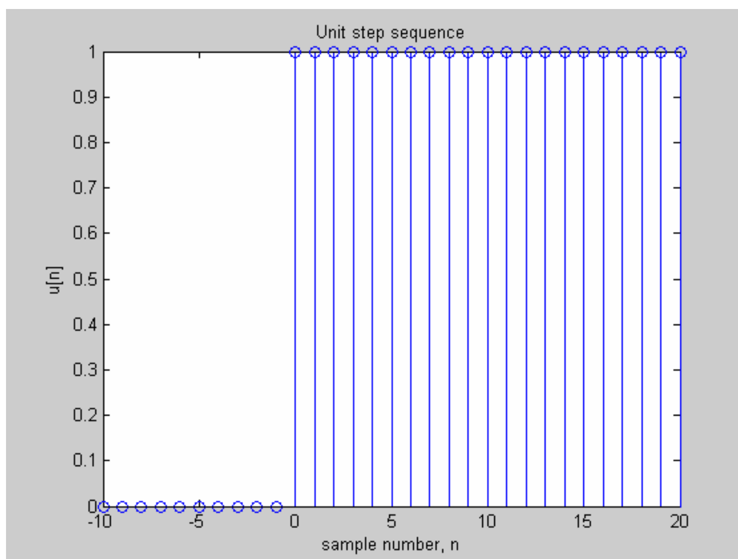
where $f[m]$ is the weight at location *n=m* and the delta function represents the unit-sample to isolate the signal at that point & only at that point.

Unit-step and generic step sequences:

$$u[n] = \begin{cases} 1 & if\ n \geq 0 \\ 0 & Otherwise \end{cases} = \sum_{m=0}^{\infty} \delta[n-m] \quad \text{and} \quad u[n-n_0] = \begin{cases} 1 & if\ n \geq n_0 \\ 0 & if\ n < n_0 \end{cases} \qquad (2.28)$$

%Example 2.4: Unit step sequence

```
n=-10:1:20;
f=zeros(n);
f(11:31)=1;
axis([-10,20,-1,2]);
stem(n,f,)
xlabel ('sample number, n');
ylabel ('u[n]');
ptitle('Unit step sequence');
```



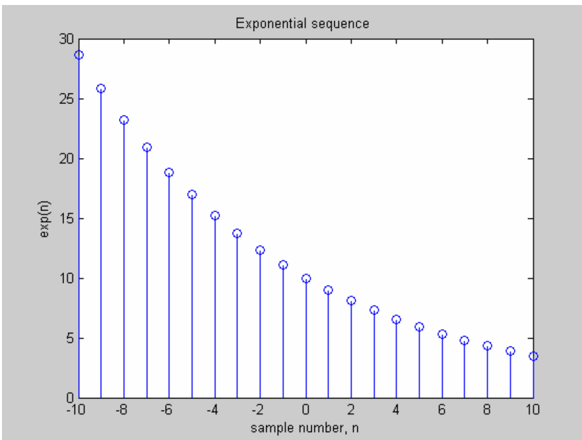
Exponential sequences (Real):

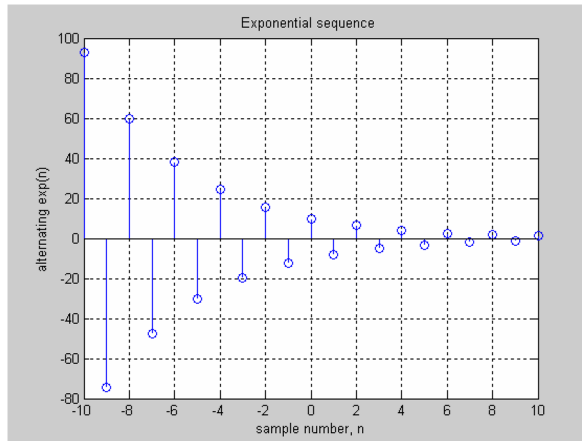
$$x[n] = A.a^n \quad where\ A = Amplitude\ and\ a = Base \qquad (2.29)$$

%Example 2.5: Decaying Exponential sequence alternating sequence

```
n=-10:1:10;
exp =10*(.9).^n;
axis([-10 10 0 30]);
stem(n,exp)
xlabel('sample number, n');
ylabel ('exp(n)');
title('Exponential sequence ')
```

```
% Alternating decaying sequence
        n=-10:1:10;
        exp2 =10*(-.8).^n;
        axis([-10 10 -30 30]);
        stem(n,exp2);
        xlabel('sample number, n'); ylabel ('alternating exp(n)');
        title('Exponential sequence ')
        grid; axis
```



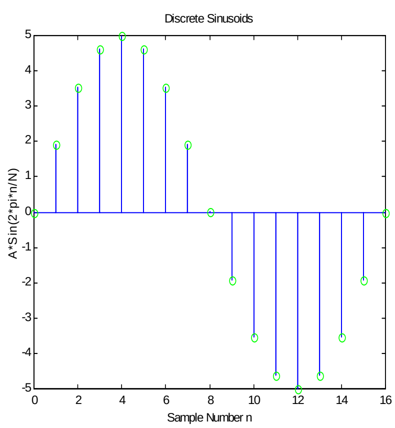


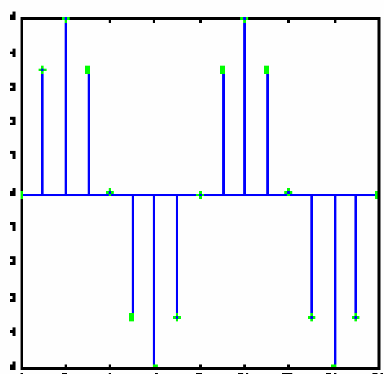
2.1.4 Discrete Sinusoidal Signal:

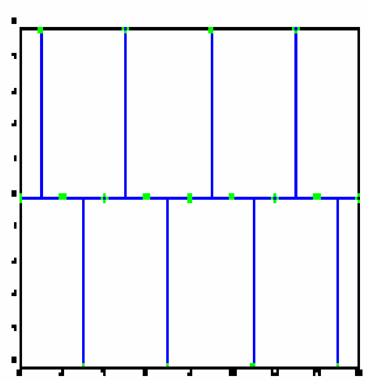
$$x[n] = A.Sin(\Omega_0 n + \varphi) = A.Sin(2\pi / Nn + \varphi) \qquad (2.30)$$

where $N = Period \quad and \quad \Omega_0 = Digital\ Fundamental\ Frequency.$

Example: 2.6: Effects of Sampling Rate on: $x[n] = A.Sin(2\pi n / N)$ where $N = 16,8,4\ Samples$.







Exponentially Modulated Sinusoidal Sequences:

$$x[n] = A.a^n.Cos(\frac{2\pi}{N}n + \theta) \qquad (2.31)$$

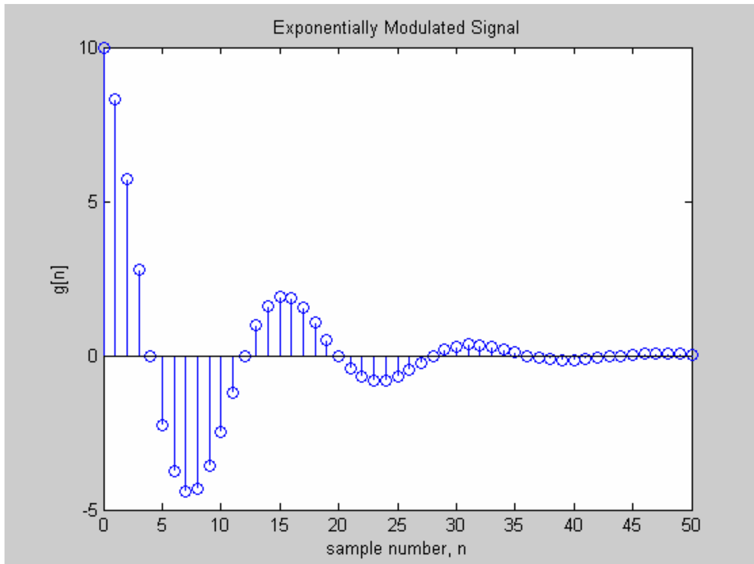
where:

$$A = Amplitude \qquad a^n = Envelope\ of\ Modulated\ Signal$$

$$Cos(\frac{2\pi}{N}n + \theta) = Oscillatory\ Component\ of\ Signal\ ;\ \theta = Initial\ Phase\ when\ n = 0$$

Example: 2.7: Exponentially modulating Sinusoidal Sequence

```
n=0:1:50;
exp=10*(0.9) .^n;   g=cos(2*n*pi/16);   h=exp .* g;
stem(n,h);
xlabel('sample number, n');  ylabel ('g[n]');  title('Exponentially Modulated Signal')
```



Sync function:

$$Sinc(\lambda) \equiv \frac{Sin(\pi\lambda)}{\pi\lambda} \tag{2.32}$$

which frequently occurs in Fourier analysis in integrals computing c_n:

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi.ft}dt = \frac{1}{j2\pi.fT_0}(e^{j2\pi.ft} - e - {}^{j2\pi.ft}) = \frac{Sin(\pi.fT_0)}{\pi.fT_0} = Sinc(\pi.fT_0) \tag{2.33}$$

and Sinc function has nulls (zero-crossings, attains value of zero, etc.) at integer multiples of λ.

Example 2.8: Let us perform the Fourier analysis of a Pulse Train (periodic pulses) with a period T_0.

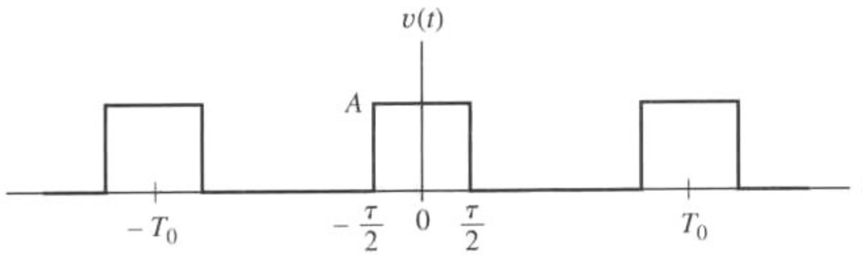


Figure 2.5 Rectangular pulse train with a period T_0. (Carlson: 2.1-7)

Since the pulse train is periodic , we can write:

$$v(t) = v(t + nT_0)$$

where

$$v(t) = \begin{cases} A & if\ |t| < a/2 \\ 0 & otherwise \end{cases} = C_0 + \sum_{n=1}^{\infty} C_n . Cos(nw_0 t + \theta_0), \tag{2.34}$$

This is valid for each and every period. The interval $[-\tau/2, \tau/2]$ is called the "ON-TIME" and the ratio: τ / T_0 is called the duty cycle of a pulse. Let us now compute the Fourier coefficients:

$$C_n = \frac{1}{T_0} \int_{-\tau/2}^{+\tau/2} A.e^{-jnw_0 t} dt = \frac{A}{T_0} . \frac{e^{-jnw_0 t}}{-jnw_0} \Bigg|_{-a/2}^{+a/2} = \frac{A}{T_0} . \frac{e^{-jnw_0 \tau/2} - e^{jnw_0 a/2}}{-jnw_0}$$

If we use the well-known trigonometric identity: $\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$ we obtain:

$$\begin{aligned} C_n &= \frac{A}{2\pi / w_0} . \frac{2}{nw_0} . Sin(nw_0 \tau / 2) = \frac{A}{\pi n} . Sin(n 2\pi f_0 \tau / 2) \\ &= Af_0 . \frac{Sin(\pi n f_0 \tau)}{\pi n f_0} = \frac{A}{T_0} Sinc(nf_0 \tau) \end{aligned} \tag{2.35}$$

Similarly, the D.C. component is computed as:

$$C_0 = \frac{A}{T_0} . \int_{-\tau/2}^{\tau/2} 1.dt = \frac{A\tau}{T_0} \tag{2.36}$$

Magnitude and phase spectrum of these terms (2.35 and 2.36) are shown in Figure 2.6 for a 25% duty cycle case: $\tau / T_0 = 1/4$..

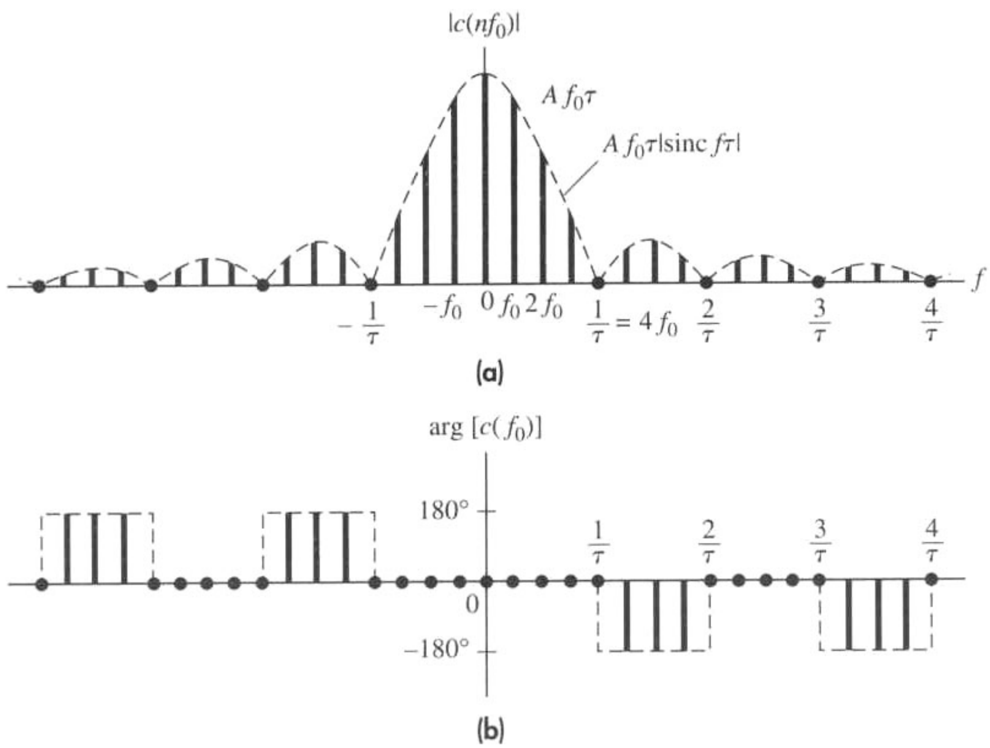


Figure 2.6 Magnitude and phase spectra of a pulse train with 25% duty cycle. (Carlson 2.1-8)

When we substitute these coefficients into the voltage equation above we obtain a series expansion representation of this pulse train function $v(t)$.

$$v(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{A}{T_0} Sinc(nf_0\tau).Cos(2\pi nf_0 t + \theta_0) \tag{2.37}$$

It is important to note that we have written a very non-linear function in the form of periodic pulses in terms of always-continuous trigonometric functions. In order this representation to be perfect the upper limit in the sum has to be infinity, which is not very practical. In many applications, a large number is used. For instance, Carlson has implemented the final result in (2.37) as shown in Figure 2.7 for N=3, 7, and 40. It is also important to note that all the coefficients are real. In other words, $\theta_n = 0$ is valid for all integer values of *n*. First few terms of this implementation (2.37) can be written as:

$$v(t) = \frac{A}{4} + \frac{\sqrt{2}A}{\pi} Cos(w_0 t + \theta_0) + \frac{A}{\pi} Cos(3w_0 t + \theta_0) + \frac{\sqrt{2}A}{3\pi} Cos(3w_0 t + \theta_0) + \cdots \tag{2.38}$$

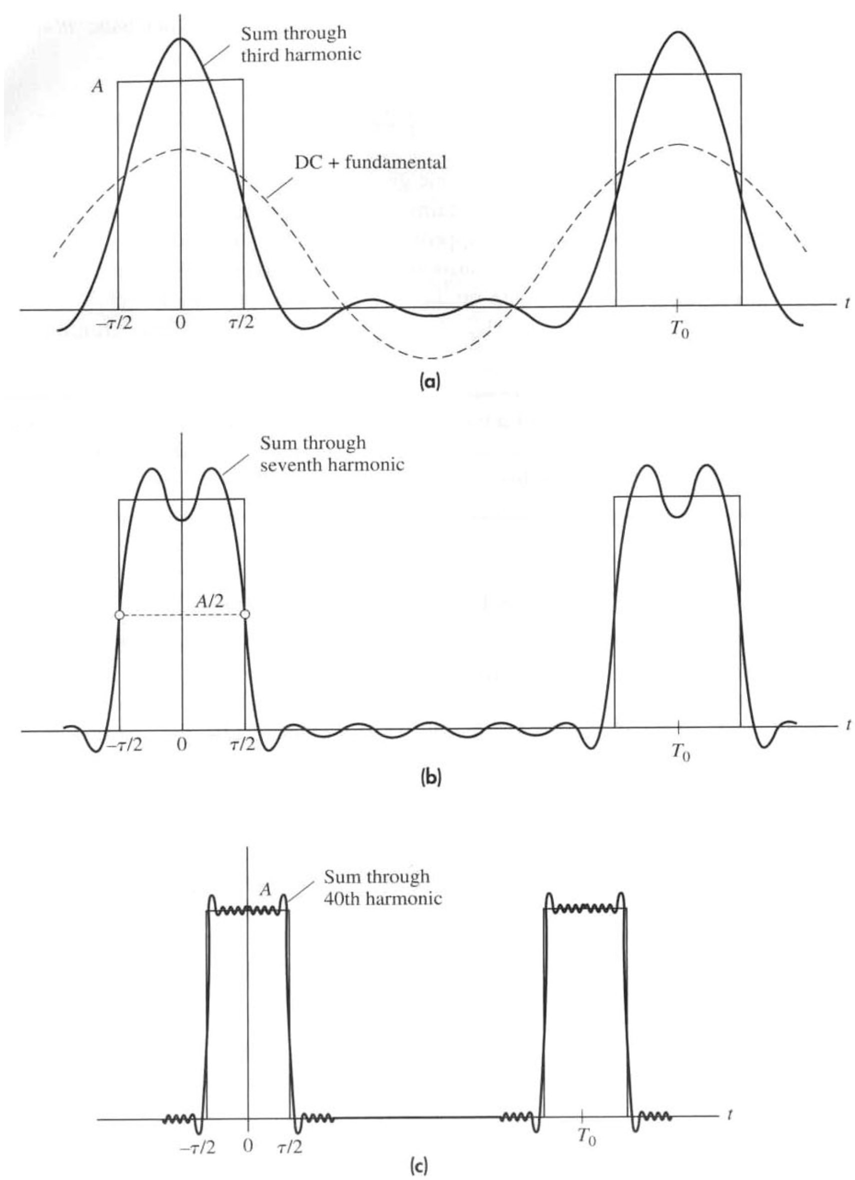


Figure 2.7 Fourier series reconstruction of a rectangular pulse train with finite terms. (Carlson 2.1-9)

2.1.5 Dirichlet Existence Conditions and Gibbs Phenomenon: Fourier analysis exists if

1. The signal is absolutely integrable with a finite value: $\int_{<T_0>} |v(t)| dt < \infty$
2. Extrema of $v(t)$ must be finite for any interval of $[t_1, t_2]$. (Minima and Maxima points)
3. Discontinuities over any finite interval of $[t_1, t_2]$ must be also finite with each discontinuity of finite size. (Differentiable)

Even if a signal satisfies the above conditions, the finite series (summation does not go to ∞), as it is the case for all real-life scenarios, has overshoots and undershoots at each point of discontinuity as seen in Figure (2.7).

2.1.6 Parseval's Power Theorem and Superposition: Average power of a periodic signal is conserved through Fourier analysis. Consider the power of a periodic signal across 1 Ohm resistor:

$$P = \frac{1}{T_0} \int_{<T_0>} |v(t)|^2 dt = \frac{1}{T_0} \int_{<T_0>} v(t).v^*(t) dt \tag{2.39}$$

Using Fourier series expansion on conjugate signal and substituting in the last equation:

$$v*(t) = [\sum_{n=-\infty}^{\infty} c_n.e^{jn2\pi f_0 t}]^* = \sum_{n=-\infty}^{\infty} c_n^*.e^{-jn2\pi f_0 t}$$

$$\begin{aligned} P &= \frac{1}{T_0} \int_{<T_0>} v(t).[\sum_{n=-\infty}^{\infty} c_n^*.e^{-jn2\pi f_0 t}] dt = \sum_{n=-\infty}^{\infty} [\frac{1}{T_0} \int_{<T_0>} v(t)..e^{-jn2\pi f_0 t} dt] c_n^* \\ &= \sum_{n=-\infty}^{\infty} c_n.c_n^* = \sum_{n=-\infty}^{\infty} | c_n |^2 \end{aligned} \tag{2.40}$$

which shows that the power is conserved (not lost) during Fourier representation and it can be computed by squaring adding the magnitudes of line spectra. The last point is simply the principle of superposition for average power.

2.2 Fourier Transform for Continuous Signals

If $v(t)$ is the voltage across a 1 Ohm resistor, the total delivered energy is found by integrating the instantaneous power:

$$E \equiv \int_{-\infty}^{\infty} | v(t) |^2 \ dt \tag{2.41}$$

if the above integral exists and results in $0 < E < \infty$ then $v(t)$ is an energy signal and it is also a non-periodic (aperiodic) signal. If it satisfies the Dirichlet conditions then it must have Fourier Transform, which is defined by:

$$V(f) = F[v(t)] = \int_{-\infty}^{\infty} v(t).e^{-j2\pi . ft} dt \tag{2.42a}$$

or in the angular frequency representation:

$$V(w) = F[v(t)] = \int_{-\infty}^{\infty} v(t).e^{-jwt} dt \tag{2.42b}$$

These forward transforms are also known as the ANALYSIS EQUATION in the community. The time function $v(t)$ can be recovered from $V(f)$ or $V(w)$ through inverse transform process and it is called SYNTHESIS EQUATION:

$$v(t) = F^{-1}[V(f)] = \int_{-\infty}^{\infty} V(f).e^{j2\pi.ft} df \qquad (2.43a)$$

and from the angular frequency representation:

$$v(t) = F^{-1}[V(w)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} V(w).e^{jwt} dw \qquad (2.43b)$$

The expressions in (2.42) and (2.43) are known as the Fourier pairs and there are tables for many frequently observed signals. For instance, there is one in Table A.2 in the appendix A of these lecture notes and in pages 780 and 781 of Carlson text.

1. Fourier transform is a complex function with the amplitude spectrum term: $|V(f)|$ and the phase spectrum: $\arg V(f)$, which implies that we need to plot two different spectra in the frequency-domain for a given time signal.
2. The value of $V(f)$ at d.c., or at $f = 0$ equals the net area under the plot of $v(t)$:

$$V(0) = \int_{-\infty}^{\infty} v(t) dt \qquad (2.44)$$

3. If $v(t)$ is real then

$$V(-f) = V^{*}(f) \qquad (2.45)$$

with properties:

$$|V(-f)| = |V(f)| \quad \text{and} \quad \arg V(-f) = -\arg V(f) \qquad (2.46)$$

amplitude spectrum has even-symmetry, while phase spectrum shows odd-symmetry.

Example 2.9: Find the Fourier transform of a single rectangular-pulse. This function is also known as a rectangular gate or a time-window function and it is shown in many ways, one of which is $\Pi(t/\tau)$ as shown in Figure 2.8.

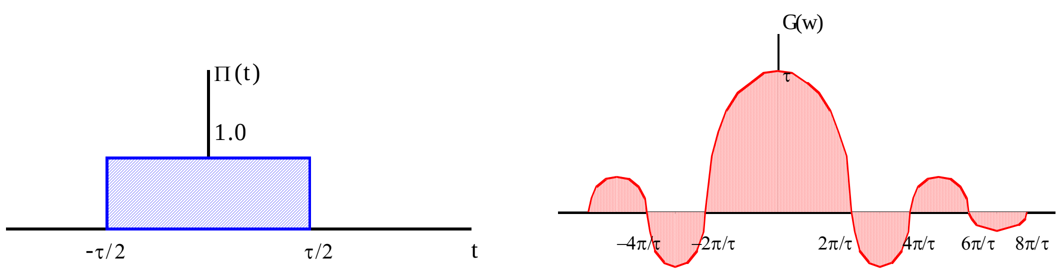


Figure 2.8 Rectangular pulse signal and its Fourier transform.

Let us compute the Fourier transform $G(w)$ of this time-window or a rectangular gate function:

$$G(w) = \int_{-\infty}^{\infty} \Pi(t/\tau).e^{-jwt} dt = \int_{-\tau/2}^{\tau/2} 1.e^{-jwt} dt = \left. \frac{e^{-jwt}}{-jw} \right|_{-\tau/2}^{\tau/2}$$

$$= \frac{1}{-jw}\left[e^{-jw\tau/2} - e^{jw\tau/2} \right] = \frac{1}{-jw}\left[-2j.Sin(w\tau/2) \right] = \frac{2}{w} Sin(w\tau/2)$$

$$= \tau.Sinc(w\tau/2\pi) = \tau.Sa(w\tau/2)$$

which is real for all values of *w*. Here *Sinc* and *Sa* represent the Sinc and sampling functions, respectively, and they are defined by:

$$Sinc(x) = \frac{Sin(\pi x)}{\pi x} \quad \text{and} \quad Sa(x) = \frac{Sin(x)}{x}$$

It is easy to see that the phase term (argument) is uniformly zero.

$$ArgG(w) = \phi(w) = 0 \;\; for \;\; all \;\; w.$$

Also, 90 percent of the energy of the total spectrum is under the *MAIN LOBE*. From the results of this example we can conclude that:

$$\Pi(t/\tau) \Leftrightarrow \tau.Sinc(w\tau/2\pi).$$

It is worth noting that the Carlson text has the same problem analyzed in terms of f and the result is given in (9b) as:

$$V(f) = A.\tau.Sinc(f\tau)$$

which is equivalent to the result above.

Example 2.10: Find the Fourier transform of a sinusoid: $x(t) = A.Cos(w_c t)$

Using the trigonometric identity: $Cos(x) = 1/2.(e^{jx} + e^{-jx})$ to rewrite our signal.

$$F\{A.Cos(w_c t)\} = \frac{A}{2}\int_{-\infty}^{\infty}(e^{jw_c t} + e^{-jw_c t}).e^{-jwt}dt = \frac{A}{2}[\int_{-\infty}^{\infty} e^{jw_c t}.e^{-jwt}dt + \int_{-\infty}^{\infty} e^{-jw_c t}.e^{-jwt}dt]$$

$$= (A/2).[2\pi\delta(w - w_c) + 2\pi\delta(w + w_c)] = A\pi[\delta(w - w_c) + \delta(w + w_c)]$$

which is a pair of impulses of height $A\pi$ located symmetrically at $\mp w_c$.

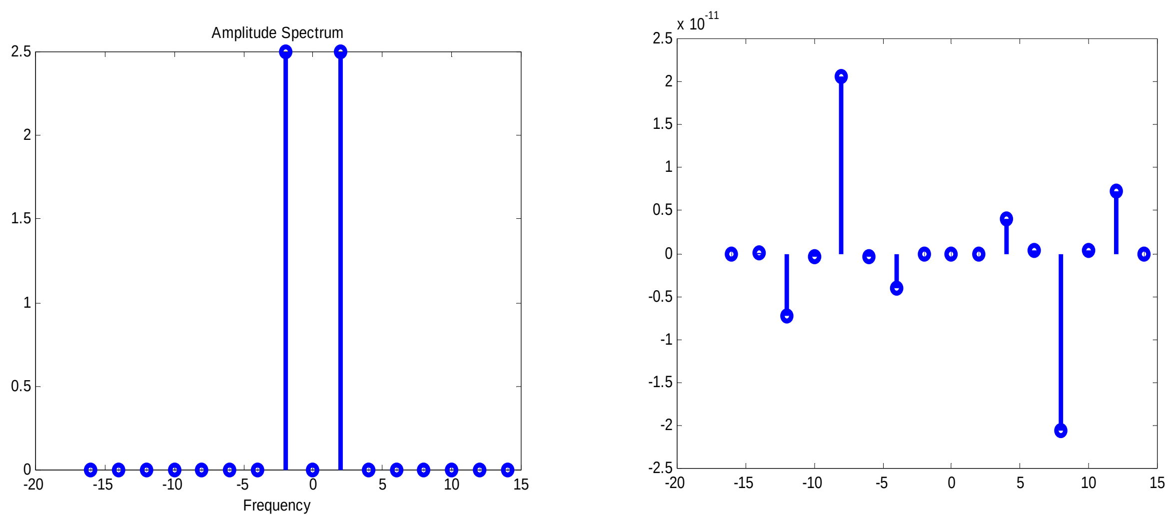


Figure 2.9 Single sinusoid and its Fourier transform

Fourier transform is sometimes expanded into its real and imaginary terms:

$$V(f) = V_e(f) + jV_o(f) \tag{2.47}$$

where

$$V_e(f) = \int_{-\infty}^{\infty} v(t).Coswt.dt \quad \text{and} \quad V_o(f) = -\int_{-\infty}^{\infty} v(t).Sinwt.dt \tag{2.48}$$

are the cosine and sine Fourier integrals. If the signal $v(t)$ is an even function in the time-domain then:

$$V(f) = V_e(f) = 2\int_{-\infty}^{\infty} v(t).Coswt.dt \quad \text{and} \quad V_o(f) = 0 \tag{2.49}$$

Similarly, for a signal $v(t)$ with an odd-symmetry in the time-domain, we have:

$$V_e(f) = 0 \quad \text{and} \quad V_o(f) = -2j.\int_{-\infty}^{\infty} v(t).Sinwt.dt \tag{2.50}$$

Example 2.11: Find the Fourier transform of a causal (one-sided) exponential pulse.

$$v(t) = \begin{cases} Ae^{-bt} & t > 0 \\ 0 & t < 0 \end{cases} \tag{2.51}$$

$$V(f) = \int_{-\infty}^{\infty} v(t).e^{-j2\pi ft}dt = \int_{0}^{\infty} Ae^{-bt}.e^{-j2\pi ft}dt = \int_{0}^{\infty} Ae^{-bt}.e^{-jst}dt \tag{2.52}$$

where the last integral is the Laplace transform of $v(t)$ with $s = j2\pi f$. We can find the above integral from Laplace or Fourier transform tables as:

$$V(f) = \frac{A}{b+s} = \frac{A}{b + j2\pi f} \tag{2.53}$$

This complex results is not useful, we can find even and odd terms or more appropriately, the amplitude and phase terms:

$$V_e(f) = RE\{V(f)\} = \frac{bA}{b^2 + (2\pi f)^2}; \qquad V_o(f) = \operatorname{Im}\{V(f)\} = -\frac{2\pi fA}{b^2 + (2\pi f)^2} \tag{2.54}$$

$$|V(f)| = \sqrt{V_e^2(f) + V_o^2(f)} = \frac{A}{\sqrt{b^2 + (2\pi f)^2}}; \quad \arg V(f) = \arctan\frac{V_o(f)}{V_e(f)} = -\arctan\frac{2\pi f}{b} \tag{2.55}$$

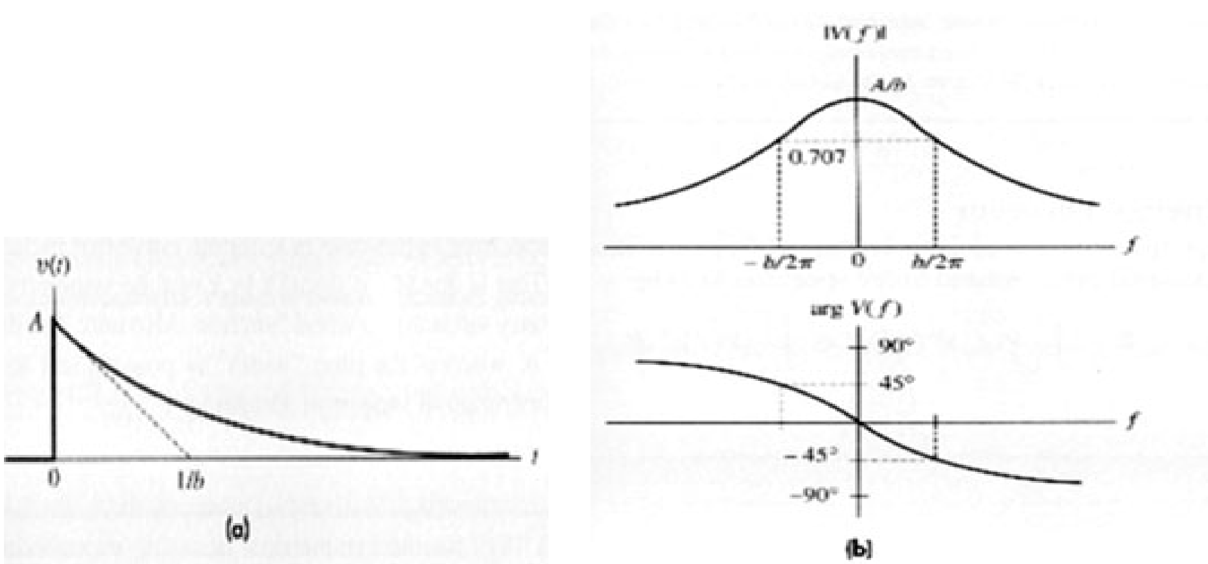


Figure 2.10 Causal exponential signal and its amplitude and phase spectra. (Carlson 2.3-3)

Rayleigh's Energy Theorem: Energy is pf a signal is related to its spectrum by:

$$E = \int_{\infty} V(f).V^*(f)dt = \int_{\infty} |V(f)|^2 df \tag{2.56}$$

and the term $|V(f)|^2$ is known as the Energy Spectral Density or the distribution of energy in the frequency domain. For instance, the energy spectral density and the energy is simply found from the result of Example 2.9:

$$\int_{-1/\tau}^{-1/\tau} |V(f)|^2\ df = \int_{-1/\tau}^{-1/\tau} (A\tau)^2.Sinc^2 f\tau.df = 0.92.A^2\tau$$

Notes:

1. The plot in that example was over $w = 2\pi f$, therefore scaling was needed and
2. Last integral is found from numerical computation since it is not available in a closed form.
3. Since the total energy in time domain was $E_t = A^2\tau$, then 90% of total energy is under the mail lobe as seen in Figure 2.11.

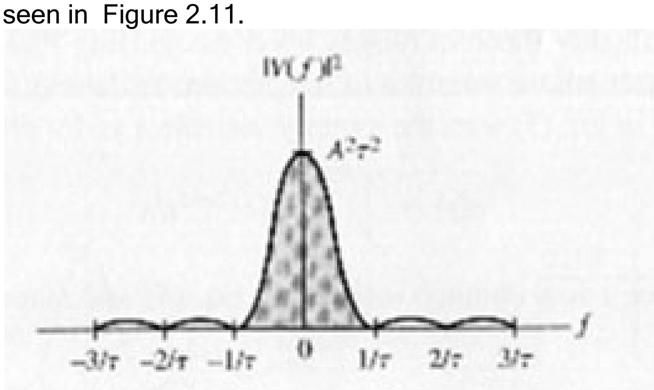


Figure 2.11 Energy spectral density of a rectangular pulse. (Carlson 2.2-4)

Duality Theorem: If $v(t) \Leftrightarrow V(f)$ are Fourier pairs and if there is a time-function $z(t)$ related to the functional form $V(f)$ by

$$z(t) = V(t) \qquad \text{then} \qquad F[z(t)] = v(-f) \tag{2.57}$$

Example 2.12: Find the Fourier transform of a sync pulse using Duality Theorem. Consider the sync pulse defined by: $z(t) = A.Sync2Wt$

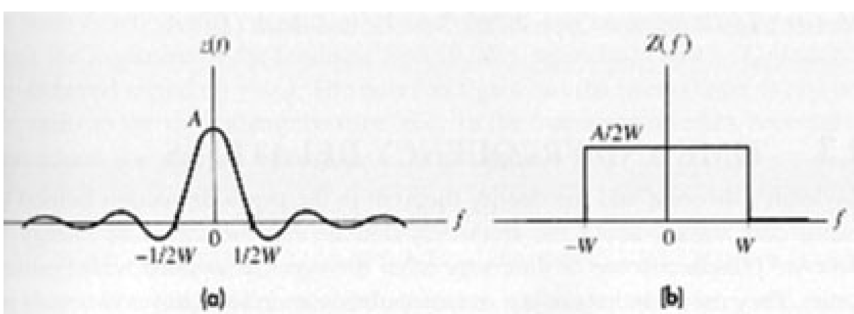


Figure 2.12 Sync pulse in the time-domain and its spectrum. (Carlson 2.2-5)

From the duality principle:

$$v(t) = \frac{A}{2W}.\Pi(t/\tau) \qquad \text{and} \qquad V(f) = \frac{A}{2W}\tau.Sinc(t.2W)$$

with $z(t) = V(t); \quad \tau = 2W$, we have:

$$F[z(t)] = v(-f) = \frac{A}{2W}\Pi(\frac{-f}{2W}) \qquad \text{and} \qquad Z(f) = \frac{A}{2W}\Pi(\frac{f}{2W}) \tag{2.58}$$

We can see that the time-unlimited sync pulse results in a band-limited spectrum in the frequency domain. There are several important functions and their Fourier transforms used in communication and signal processing which are listed in Appendix B of these lecture notes.

2.3 Time and Frequency Relations

2.3.1 Superposition Theorem: If $\{a_1, a_2\}$ are a pair of constants and

$$v(t) = a_1.v_1(t) + a_2.v_2(t) \qquad \text{then} \qquad V(f) = a_1.V_1(t) + a_2.V_2(t) \tag{2.59}$$

2.3.2 Time Delay: If a signal $v(t)$ is delayed by t_d in a system to yield $v(t - t_d)$, then in the frequency-domain the phase spectrum is shifted by a linear phase with a slope $-2\pi t_d$ to result:

$$v(t - t_d) \Leftrightarrow V(f).e^{-j2\pi f.t_d} \tag{2.60}$$

Note that the amplitude spectrum is NOT changed through time-delay due to the fact that:

$$|V(f).e^{-j2\pi f.t_d}| = |V(f)|.|e^{-j2\pi f.t_d}| = |V(f)|.1 = |V(f)| \tag{2.61}$$

2.3.3 Scale change in time-domain is equivalent to inverse scaling in the frequency-domain:

$$v(at) \Leftrightarrow \frac{1}{|a|}.V(\frac{f}{a}) \quad a \neq 0 \tag{2.62}$$

Depending upon whether $|a| > 1$ or $|a| < 1$ the process could result in compression or expansion in time domain and the reverse occurs in the frequency-domain.

2.3.4 Frequency Translation and Modulation is the dual of the time-delay result of (2.60). That is:

$$V(f - f_c) \Leftrightarrow v(t).e^{-j2\pi f_c.t} \tag{2.63}$$

As it is apparent from above, shifting the information from its original frequency range to the neighborhood of a frequency f_c (usually called carrier frequency) is equivalent to multiplying a signal in the time domain by an exponential factor .

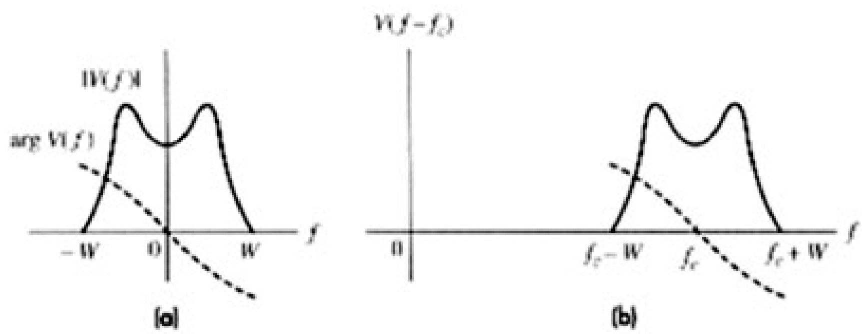


Figure 2.13 Frequency-translation (Modulation) of a bandlimited signal. (Carlson 2.3-2)

More important form is the combination of two complex exponentials (multiplication by a cosine function) is used and known as the modulation theorem:

$$v(t).Cos(2\pi f_c t + \phi) \Leftrightarrow \frac{e^{j\phi}}{2}V(f - f_c) + \frac{e^{-j\phi}}{2}V(f + f_c) \tag{2.64}$$

In the case of modulation used in radio communication, it is common to use $\phi = 0$, which results in the well-known result:

$$v(t).Cos(2\pi f_c t) \Leftrightarrow 1/2.V(f-f_c)+1/2.V(f+f_c) \quad (2.65)$$

so the original spectrum is split equally to the neighborhoods of f_c and $-f_c$. The image in the neighborhood of negative frequency $-f_c$ is known as the mirror image spectrum and cannot be seen in the measurements but it is known to be out there.

Example 2.13: Find the spectrum of an RF pulse, which is formed as the product of a gate function with a sinusoid.

$$z(t) = A.\Pi(\frac{t}{\tau}).Cos(2\pi w_c.t) \text{ then } Z(f) = \frac{A\tau}{2}.Sinc(f-f_c)\tau + \frac{A\tau}{2}.Sinc(f+f_c)\tau \quad (2.66)$$

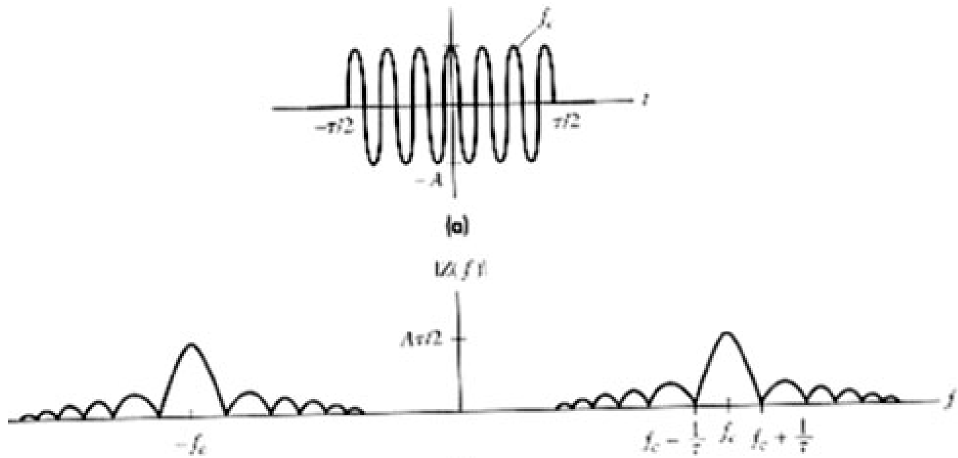


Figure 2.14. RF Pulse and it is frequency-translated (modulated) spectrum.

2.3.5 Differentiation and Integration in Time-Domain: Consider a time-domain signal $v(t)$, then its differentials and integrals exhibit the following properties:

$$\frac{d^n}{dt^n}v(t) \Leftrightarrow (j2\pi f)^n.V(f) \quad (2.67)$$

$$\int_{-\infty}^{t} v(\lambda)d\lambda \Leftrightarrow \frac{1}{j2\pi f}.V(f) \quad (2.68)$$

2.4 Convolution

Convolution Integral: Consider a signal $v(t)$ is processed by a system with an impulse response $w(t)$ then the output $y(t)$ is obtained by convolving these two functions in the time-domain:

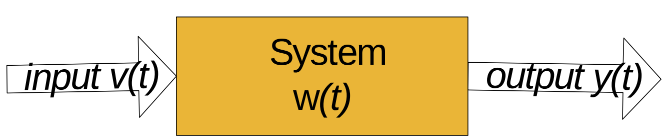


Figure 2.15 Convolution operation block diagram.

$$y(t) = v(t) * w(t) = \int_{-\infty}^{\infty} v(\tau).w(t-\tau)d\tau = \int_{-\infty}^{\infty} w(\tau).v(t-\tau)d\tau \quad (2.69)$$

It is worth noting that the convolution operation is commutative, i.e., the roles of the signal $v(t)$ and the system $w(t)$ function can be interchanged.

Similarly, it can be shown that convolution is also associative and distributive. In other words,

$$v(t)*[w(t)*z(t)] = [v(t)*w(t)]*z(t) \quad Associative \tag{2.70}$$

$$v(t)*[w(t)+z(t)] = [v(t)*w(t)]+[v(t)*z(t)] \quad Distributive \tag{2.71}$$

Example 2.14: Convolution of a causal exponential function and a ramp function as shown in Figure 2.16.

$v(t) = A.e^{-t}$ if $0 < t < \infty$ and $w(t) = t/T$ if $0 < t < T$

$w(t-\lambda) = (t-\lambda)/T$ if $0 < t-\lambda < T$

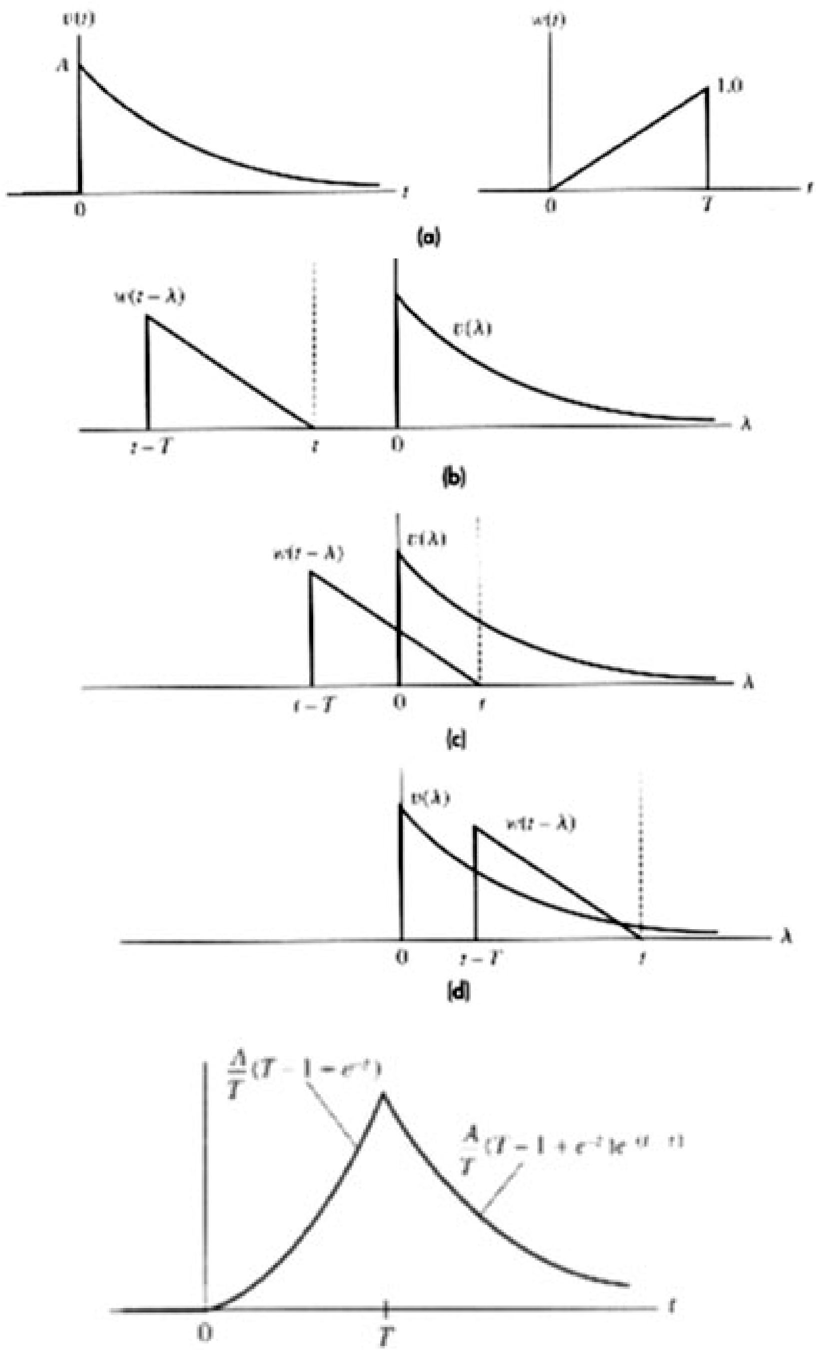


Figure 2.16 Demonstration of convolution operation. (Carlson 2.4-1)

Case 1: $t<0$: In this case, there is no overlap between the two functions, therefore,

$$y(t)=v(t)*w(t)=0 \quad if\ t<0$$

Case 2: $0<t<T$:

$$y(t)=v(t)*w(t)=\int_0^t A.e^{-\lambda}.(\frac{t-\lambda}{T}).d\lambda=\frac{A}{T}.t.\int_0^t e^{-\lambda}d\lambda-\frac{A}{T}.\int_0^t \lambda.e^{-\lambda}d\lambda$$

$$=(A/T).(t-1+e^{-t}) \quad if\ 0<t<T$$

The last set of integrals can be found in the integration tables and in the Appendix A.5 of the lecture notes.

Case 3: $t>T$:

$$y(t)=v(t)*w(t)=\int_{t-T}^t A.e^{-\lambda}.(\frac{t-\lambda}{T}).d\lambda=\frac{A}{T}.t.\int_{t-T}^t e^{-\lambda}d\lambda-\frac{A}{T}.\int_{t-T}^t \lambda.e^{-\lambda}d\lambda$$

$$=(A/T).(T-1+e^{-T}).e^{-(t-T)} \quad if\ t>T$$

Table 1 Continuous Fourier Transform and Convolution Properties			
	Property	Time-Domain	Frequency-Domain
1	Superposition	$a.x_1(t)+b.x_2(t)$	$a.X_1(w)+b.X_2(w)$
2	Time Delay	$x(t-\tau_0)$	$X(w).e^{-jw\tau_0}$
3	Frequency-Translation	$x(t).e^{jw_0t}$	$X(w-w_0)$
4	Scale Change	$x(kt)$	$1/\|k\|.X(w/k)$
5	Time-Differentiation	$\frac{d^n}{dt^n}x(t)$	$(jw)^n.X(w)$
6	Time-Integration	$\int_{-\infty}^t x(\alpha).d\alpha$	$\frac{1}{jw}.X(w)+\pi.X(0).\delta(w)$
7	Duality	$X(t)$	$2\pi.x(-w)$
8	Parseval's Theorem	$\int_{-\infty}^{\infty} x(t).x^*(t)dt=\int_{-\infty}^{\infty}\|x(t)\|^2dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}X(w).X^*(w)dw=\frac{1}{2\pi}\int_{-\infty}^{\infty}\|X(w)\|^2dw$
9	Time-Convolution	$x(t)*h(t)=h(t)*x(t)$	$X(w).H(w)=H(w).X(w)$
10	Time-Multiplication	$x(t).p(t)$	$1/2\pi.X(w)*P(w)$
11	Analog Modulation	$x(t).Cos(W_ct)$	$1/2.[X(w-W_0)+X(w+W_0)]$

$$F[v(t)*w(t)]=\int_{-\infty}^{\infty}[\int_{-\infty}^{\infty}v(\lambda).w(t-\lambda)d\lambda]e^{-jwt}.dt=\int_{-\infty}^{\infty}v(\lambda).[\int_{-\infty}^{\infty}w(t-\lambda)e^{-jwt}dt]d\lambda$$

$$F[v(t)*w(t)]=\int_{-\infty}^{\infty}v(\lambda).[W(f)e^{-w\lambda}]d\lambda=[\int_{-\infty}^{\infty}v(\lambda)e^{-jw\lambda}d\lambda].W(f)=V(f).W(f)=(1/2\pi).V(w).W(w)$$

which is the expected result #10 in the table above.

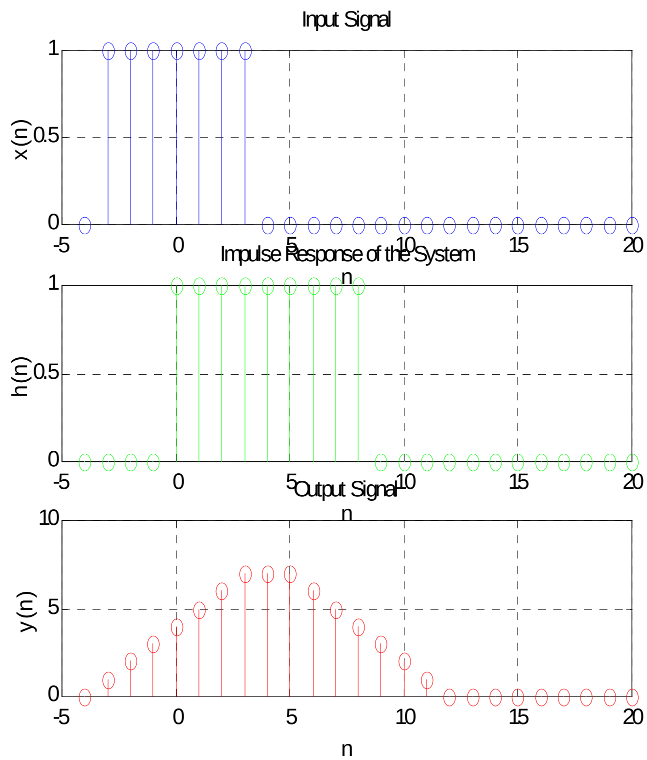
Example 2.15. Numerical convolution of two time-functions of different length.

```
% Convolution of two-discrete finite pulses
n=-4:20;
x=zeros(size(n));
h=zeros(size(n));
x(2:8)=ones(size(n(2:8)));
h(5:13)=ones(size(n(5:13)));

%Plot of the input and system
axis([-4,20,0,2])
subplot(311),stem(n,x);
grid;
title('Input Signal');
xlabel('n');
ylabel('x(n)');
subplot(312),stem(n,h,'g');
grid;
xlabel('n');
ylabel('h(n)');

title('Impulse Response of the System');

%convolution
y=conv(h,x);
subplot(313),stem(n,y(5:29),'r');grid;
title('Output Signal');
xlabel('n'); ylabel('y(n)');
```



Example 2.16: Let us use this theorem to find the impulse response and the unit-step response of a system $h(t)$: For the case of impulse response we assume that the input is an impulse function: $x(t) = \delta(t)$. Then from the definition of the convolution operation we write:

$$y(t) = x(t) * h(t) = \delta(t) * h(t) = \int_{-\infty}^{\infty} \delta(\tau).h(t-\tau)d\tau = h(t) \tag{2.71}$$

The last integral above is obtained from the *Sifting Theorem* discussed earlier. It is important to note that the term "impulse response" gets its name from pushing a unit-impulse through a system and measuring its response. Similarly, the unit-step response is computed by using a unit-step signal $u(t)$ as input:

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau).u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau \tag{2.72}$$

It is not difficult to see the convolution of a signal by a unit-step $u(t)$ is equivalent to passing the signal through a perfect integrator.

Example 2.17: Ideal Low-pass Filter. Consider a rectangular signal $v(t) = A.\Pi(t/\tau)$. Its Fourier transform is a sync function: $V(f) = A.\tau.Sinc(f\tau)$, which has tails all the way to infinity in both directions. Let us multiply this $V(f)$ by a rectangular function (ideal low-pass filter) in the frequency-domain:

$$W(f) = \Pi(\frac{f}{2/\tau}) \Leftrightarrow w(t) = \frac{2}{\tau}.Sinc(\frac{2t}{\tau}) \quad \Rightarrow z(t) = v(t) * w(t)$$

$$Z(f) = V(f).W(f) \Rightarrow \quad z(t) = \int_{t-\frac{\tau}{2}}^{t+\frac{\tau}{2}} \frac{2A}{\tau}.Sinc(\frac{2\lambda}{\tau}).d\lambda$$

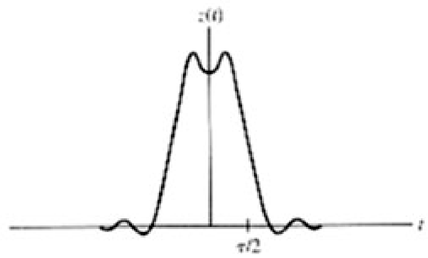
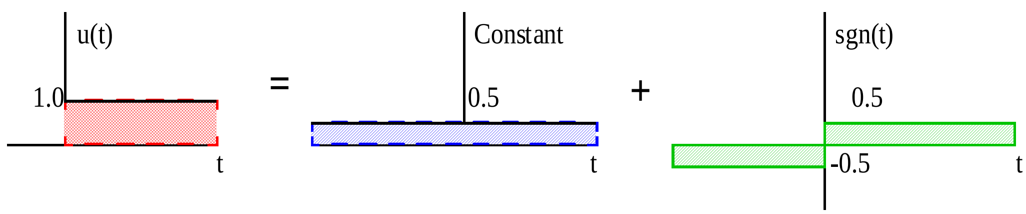


Figure 2.16 Output of an ideal low-pass filter to a sync signal.

Fourier Analysis of Impulse and Pulse Functions: Consider an impulse function in time domain (Dirac delta function) $\delta(t)$:

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t-\tau).e^{-jw\tau}.d\tau = e^{-jw.0} = 1 \qquad (2.73)$$

Fourier transform of a unit-step function. As we can see from the sketch below, we can write our unit-step function as a sum of two simple functions: $u(t) = \frac{1}{2} + \frac{1}{2}.Sgn(t)$



Here $Sgn(t)$ is the sign indicator function. But we can also write the derivative of this expression:

$$\frac{d}{dt}\{1/2.Sgn(t)\} = \delta(t) \quad \text{or equivalently:} \quad (jw).F[1/2.Sgn(t)] = 1,$$

which results in a form:

$$F[1/2.Sgn(t)] = 1/j2\pi f \quad \text{and} \quad F[1/2] = (1/2).2\pi\delta(w) = \pi\delta(w) = (1/2).\delta(f)$$

Finally, we combine these two transforms to obtain:

$$F[u(t)] = \frac{1}{2}.\delta(f) + \frac{1}{j2\pi f} \qquad (2.74)$$

Similarly, we can compute the following results:

If $v(t) = A \quad \Rightarrow \quad F[v(t)] = F[A] = A.\delta(f)$ (2.75)

If $v(t) = A.e^{jw_c t} \quad \Rightarrow \quad F[v(t)] = F[A.e^{jw_c t}] = A.\delta(f - f_c)$ (2.76)

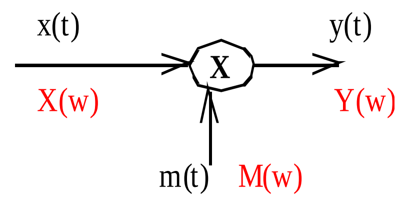
If $v(t) = A.Cosw_c t \quad \Rightarrow \quad F[A.Cosw_c t] = F[\frac{A}{2}.(e^{jw_c t} + e^{-jw_c t}] = \frac{A}{2}.\{\delta(f - f_c) + \delta(f - f_c)\}$ (2.77)

Fourier Transform pairs of many frequently used signals are given in Table A.2 of the appendices of these lecture notes.

2.5 Modulation Theorem

It is used for translating the information from the base-band frequencies to a band-pass region for efficient communication purposes and it is achieved by the multiplication of the information signal $x(t)$ with a carrier signal $m(t)$:

$$y(t) = x(t).m(t) \qquad (2.78a)$$



It is not difficult to see from Fourier properties in Table A.1, this is equivalent to a convolution operation in the frequency-domain:

$$Y(w) = (1/2\pi).X(w) * M(w) \qquad (2.78b)$$

This can be rewritten explicitly as a Fourier Integral:

$$Y(w) = \frac{1}{2\pi}\int_{\infty} M(\upsilon)X(w-\upsilon)d\upsilon = \frac{1}{2\pi}\int_{\infty} M(w-\upsilon)X(w)d\upsilon \qquad (2.79)$$

Example 2.18: Given a base-band signal with a generic spectrum as shown. Let us modulate it with an ideal impulse train and investigate what happens to the spectrum after modulation. In the t-domain the output of the modulator (mixer) is the product: $S(t) = x(t).p_{T_0}(t)$, where the input signal is $x(t)$ and the modulating impulse train $p_{T_0}(t) = \sum_{\infty} \delta(t - nT_0)$ acts as a sampling function to yield:

$$S(t) = \sum_{\infty} x(t)\delta(t - nT_0) \qquad (2.80)$$

However, the frequency-domain interpretation of this process is much more informative. To see that let us first take the Fourier transform of the ideal impulse train:

$$P_{T_0}(w) = F\{p_{T_0}(t)\} = F\{\sum_{\infty}\delta(t-nT_0)\} = \frac{2\pi}{T_0}\sum_{\infty}\delta(w - \frac{2\pi k}{T_0}) = W_0\sum_{\infty}\delta(w - kW_0)$$

and the sampled-output spectrum:

$$S(w) = \frac{1}{2\pi}.\frac{2\pi}{T_0}.\sum_{k=-\infty}^{\infty} X(w) * \delta(w - \frac{2\pi}{T_0}k) = \frac{1}{T_0}\sum_{k=-\infty}^{\infty} X(w - kW_0) \qquad (2.81)$$

$S(w)$ is a periodic replica of the baseband spectrum $X(w)$.

Example 2.19: Given a generic baseband signal with a spectrum as shown and we modulate it with a cosine signal (sinusoidal oscillation). Now let us study what happens to the spectrum after modulation.

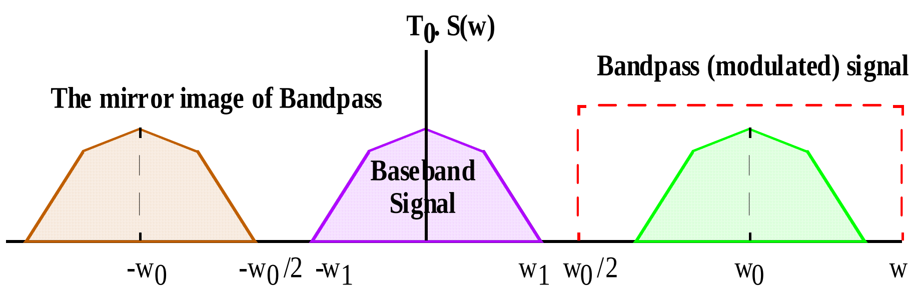


Figure 2.17 Generic baseband signal in frequency-domain and its translations.

$$y(t) = x(t).\cos(w_0 t + \theta_0) \quad \Leftrightarrow \quad (1/2).[X(w-w_0)e^{j\theta_0} + X(w+w_0)e^{-j\theta_0}] \qquad (2.82)$$

Let us consider the special case for $\theta_0 = -\pi/2$ then we have:

$$y(t) = x(t).\sin w_0 t \quad \Leftrightarrow \quad (1/2).[X(w-w_0)e^{-j\pi/2} + X(w+w_0)e^{j\pi/2}] \qquad (2.83)$$

We note that the only difference between these two cases is a phase shift (delay) of $\pi/2$.

2.6 Discrete-Time Fourier Transform and its Inverse

2.6.1 Forward DTFT: The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable w, namely:

$$X(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}, \quad w \in \Re \tag{2.84}$$

- Note n is a discrete-time instant, but w represent the continuous real-valued frequency as in the continuous Fourier transform. This is also known as the analysis equation.
- In general $X(w) \in C$
- $X(w+2n\pi) = X(w) \quad \Rightarrow w \in \{-\pi, \pi\}$ is sufficient to describe everything.
- $X(w)$ is normally called the spectrum of $x[n]$ with:

$$X(w) = |X(w)|.e^{j\angle X(w)} \quad \Rightarrow \begin{cases} |X(w)|: Magnitude\ Spectrum \\ \angle X(w): Phase\ Spectrum,\ angle \end{cases} \tag{2.85}$$

- The magnitude spectrum is almost all the time expressed in decibels (dB):

$$|X(w)|_{dB} = 20.\log_{10}|X(w)| \tag{2.86}$$

2.6.2 Inverse DTFT: Let $X(w)$ be the DTFT of $x[n]$. Then its inverse is inverse Fourier integral of $X(w)$ in the interval $\{-\pi, \pi)$.

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)e^{jwn}dw \tag{2.87}$$

This is also called the synthesis equation.

2.6.3 Convergence of DTFT: In order DTFT to exist, the series $\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$ must converge. In other words:

$$X_M(w) = \sum_{n=-M}^{M} x[n]e^{-jwn} \text{ must converge to a limit } X(w) \text{ as } M \to \infty. \tag{2.88}$$

Convergence of $X_m(w)$ for three difference signal types have to be studied:

- Absolutely summable signals: $x[n]$ is absolutely summable iff $\sum_{n=-\infty}^{\infty}|x[n]| < \infty$. In this case, $X(w)$ always exists because:

$$\left|\sum_{n=-\infty}^{\infty} x[n]e^{-jwn}\right| \le \sum_{n=-\infty}^{\infty}|x[n]|.|e^{-jwn}| = \sum_{n=-\infty}^{\infty}|x[n]| < \infty \tag{2.89}$$

- Energy signals: Remember $x[n]$ is an energy signal iff $E_x \equiv \sum_{n=-\infty}^{\infty}|x[n]|^2 < \infty$. We can show that $X_M(w)$ converges in the *mean-square* sense to $X(w)$:

$$\lim_{M \to \infty}\int_{-\pi}^{\pi}|X(w) - X_M(w)|^2 dw = 0 \tag{2.90}$$

Note that mean-square sense convergence is weaker than the uniform (always) convergence of (2.78).

- Power signals: $x[n]$ is a power signal iff

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 < \infty \qquad (2.91)$$

- In this case, $x[n]$ with a finite power is expected to have infinite energy. But $X_M(w)$ may still converge to $X(w)$ and have DTFT.
- Examples with DTFT are: periodic signals and unit step-functions.
- $X(w)$ typically contains continuous delta functions in the variable w.

2.6.4 DTFT Examples:

Example 2.20 Find the DTFT of a unit-sample $x[n] = \delta[n]$.

$$X(w) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{\infty} \delta[n]e^{-jwn} = e^{-j0} = 1 \qquad (2.92)$$

Similarly, the DTFT of a generic unit-sample is given by:

$$DTFT\{\delta[n-n_0]\} = \sum_{n=-\infty}^{\infty} \delta[n-n_0]e^{-jwn} = e^{-jwn_0} \qquad (2.93)$$

Example 2.21 Find the DTFT of an arbitrary finite duration discrete pulse signal in the interval: $N_1 < N_2$:

$$x[n] = \sum_{k=-N_1}^{N_2} c_k \delta[n-k]$$

Note: $x[n]$ is absolutely summable and DTFT exists:

$$X(w) = \sum_{n=-\infty}^{\infty} \{ \sum_{k=-N_1}^{N_2} c_k \delta[n-k]\} e^{-jwn} = \sum_{k=-N_1}^{N_2} c_k \{ \sum_{n=-\infty}^{\infty} \delta[n-k]e^{-jwn}\} = \sum_{k=-N_1}^{N_2} c_k e^{-jwk} \qquad (2.94)$$

Example 2.22 Find the DTFT of an exponential sequence: $x[n] = a^n u[n] \quad where\,|a|<1$. It is not difficult to see that this signal is absolutely summable and the DTFT must exist.

$$X(w) = \sum_{n=-\infty}^{\infty} a^n .u[n]e^{-jwn} = \sum_{n=0}^{\infty} a^n .e^{-jwn} = \sum_{n=0}^{\infty} (ae^{-jw})^n = \frac{1}{1-ae^{-jw}} \qquad (2.95)$$

Plot of the magnitude spectrum for DTFT and $X_M(w)$ for: $a = 0.8$ and $M = \{2,5,10,20,\infty = DTFT\}$ is illustrated in Figure 2.18

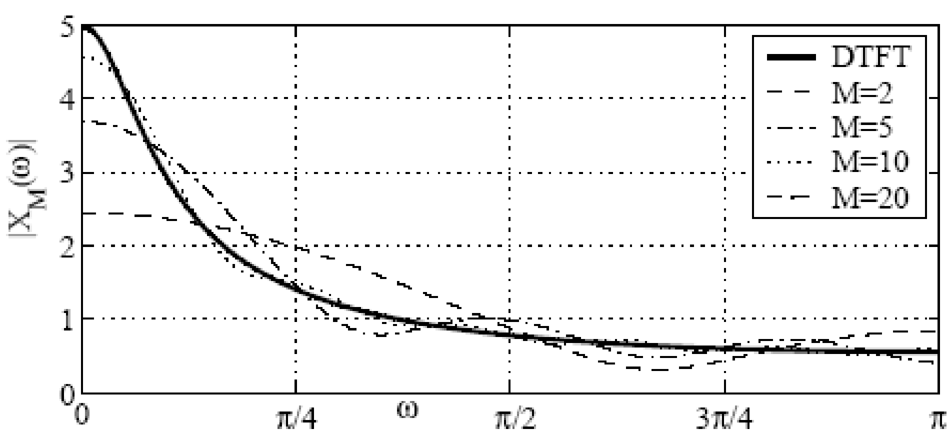


Figure 2.18 Magnitude spectrum for DTFT

2.7 Discrete Fourier Transform and its Inverse

2.7.1 DFT: It is a transformation that maps an *N*-point Discrete-time (DT) signal *x*[*n*] into a function of the N complex discrete harmonics. That is, given $x[n]; \quad n = 0,1,2,\cdots,N-1$, an *N*-point Discrete-time signal *x*[*n*] then DFT is given by (analysis equation):

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} \qquad for \quad k = 0,1,2,\cdots,N-1 \tag{2.96}$$

and the inverse DFT (**IDFT**) is given by (synthesis equation):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{+j\frac{2\pi}{N}nk} \qquad for \quad n = 0,1,2,\cdots,N-1 \tag{2.97}$$

Notes:

1. These two equations form DFT pair.
2. They have both N-point resolution both in the discrete-time domain and discrete-frequency domain.
3. Always the scaling factor $1/N$ is associated with the synthesis equation (inverse DFT).
4. $X(k)$ is periodic in N or equivalently in $\Omega_k = 2\pi/N$; that is

$$X(k) = X(\Omega_k) = X(\Omega_k + 2\pi) = X(\frac{2\pi}{N}(k+N)) = X(k+N) \tag{2.98}$$

5. *x*[*n*] determined from (2.86) is also periodic in N as shown in Figure 2.19: $x[n] = x[n+N]$

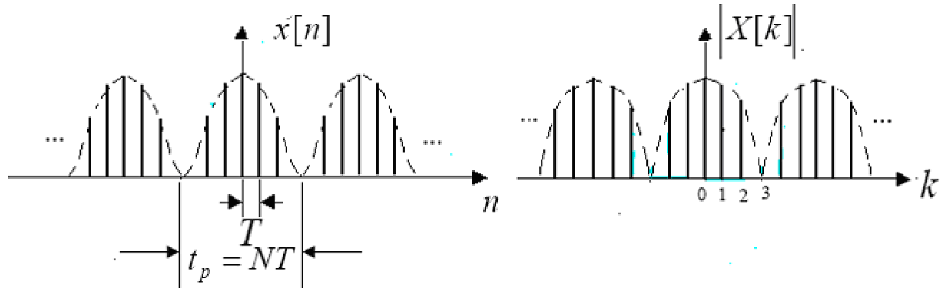


Figure 2.19 Periodic behavior of DFT.

2.7.2 DFT Examples:

Example 2.23: Compute DFT of the following two sequences: $h[n] = \{1,3,-1,-2\}$ and $x[n] = \{1,2,0,-1\}$

Note: $N = 4 \quad \Rightarrow \quad e^{j2\pi/N} = e^{j2\pi/4} = e^{j\pi/2} = j$

Let us use this information in (2.98) to compute DFT values: $H(k) = \sum_{n=0}^{3} h[n] e^{-j\frac{\pi}{2}nk} \qquad for \quad k = 0,1,2,3$

$$H(0) = h[0] + h[1] + h[2] + h[3] = 1$$

$$H(1) = h[0] + h[1]e^{-j\pi/2} + h[2].e^{-j\pi} + h[3].e^{-j3\pi/2} = 2 - j5$$

$$H(2) = h[0] + h[1]e^{-j\pi} + h[2].e^{-j2\pi} + h[3].e^{-j3\pi} = -1$$

$$H(3) = h[0] + h[1]e^{-j3\pi/2} + h[2].e^{-j3\pi} + h[3].e^{-j9\pi/2} = 2 + j5$$

Similarly,

$$X(0) = x[0] + x[1] + x[2] + x[3] = 2$$

$$X(1) = x[0] + x[1]e^{-j\pi/2} + x[2].e^{-j\pi} + x[3].e^{-j3\pi/2} = 1 - j3$$

$$X(2) = x[0] + x[1]e^{-j\pi} + x[2].e^{-j2\pi} + x[3].e^{-j3\pi} = 0$$

$$X(3) = x[0] + x[1]e^{-j3\pi/2} + x[2].e^{-j3\pi} + x[3].e^{-j9\pi/2} = 1 + j3$$

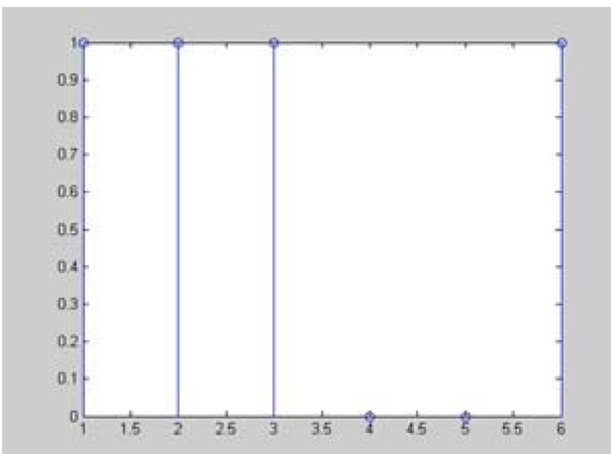
- Watch for the conjugate symmetry of terms; i.e., complex harmonics come in pairs.

Example 2.24: Given a discrete-time pulse signal $x[nT] = u[(n+1)T] - u[(n-3)T] \quad where\ T = 0.2\ s.$ (a) Use a six-point DFT to compute $X(k)$. (b) Compute the IDFT of $X(k)$.

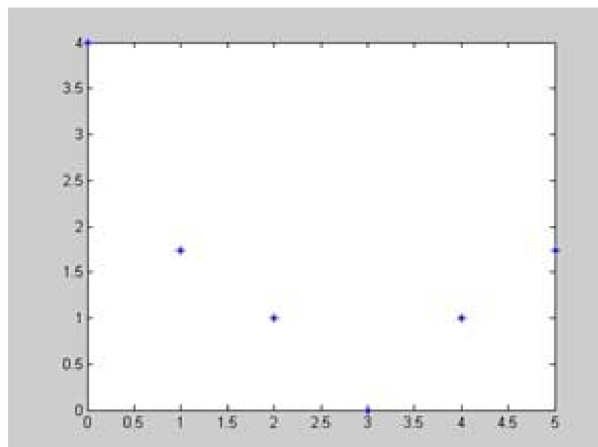
Let us start the samples at $t = -0.2$ then the six samples of the periodic extension would be $x[n] = [\overline{1},1,1,0,0,1]$. Then the script is simply:

% Example 2.33
% Part (a)
x=[1,1,1,0,0,1]; N=size(x,2); T=0.2;
stem(x); X=fft(x); disp(X);
	Answers>> Columns 1 through 4
	4.0000 1.5000 - 0.8660i -0.5000 + 0.8660i 0
	Columns 5 through 6
	-0.5000 - 0.8660i 1.5000 + 0.8660i

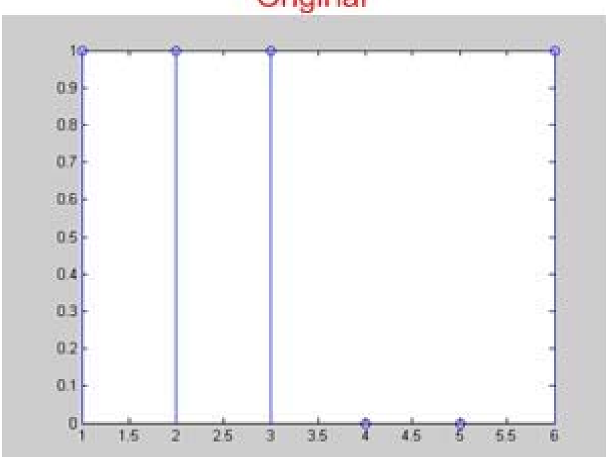
Mag=abs(X); Phase=angle(X);
% Plots; figure;
plot(n,Mag,'*'); figure;
plot(n,Phase,'+');
	% Part (b)
	xr=ifft(X);
	figure;
	stem(xr);



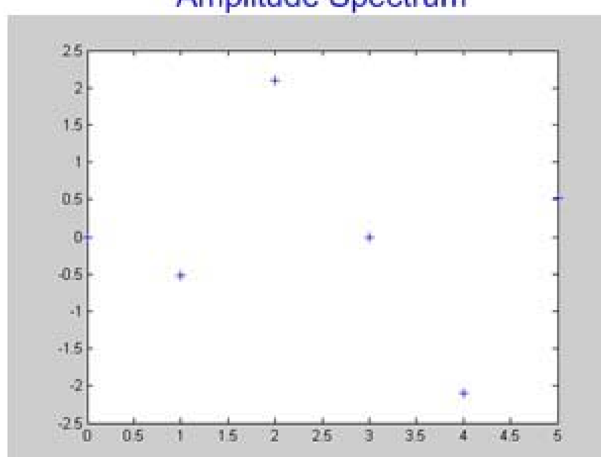
Original



Amplitude Spectrum



Reconstructed



Phase Spectrum

Input signal is exactly recovered by means of a full DFT and IDFT process.