Appendix A: Mathematical Facts and Fourier Analysis

A.1. Elementary Complex Functions:

$$e^{p/2} = +j \qquad e^{-p/2} = -j \qquad e^{\pm jnp} = \begin{cases} 1 & if\ n\ even \\ -1 & if\ n\ odd \end{cases}$$

$$a + jb = r.e^{jq} \qquad r = \sqrt{a^2 + b^2} \qquad q = \arctan(b/a)$$

$$e^{iq} = Cosq + j.Sinq \qquad e^{-iq} = Cosq - j.Sinq$$

A.2. Trigonometric Identities:

$$Sin2x = 2.Sinx.Cosx \qquad Cos2x = Cos^2x - Sin^2x$$

$$Sin(x \pm p/2) = \pm Cosx \qquad Cos(x \pm p/2) = \mp Sinx$$

$$e^{jx} = Cosx + j.Sinx \qquad e^{-jx} = Cosx - j.Sinx$$

$$Sinx = \frac{1}{2j}.(e^{jx} - e^{-jx}) \qquad Cosx = \frac{1}{2}.(e^{jx} + e^{-jx})$$

$$Sin^2x + Cos^2x = 1$$

$$Sin^2x = \frac{1}{2}.(1 - Cos2x) \qquad Cos^2x = \frac{1}{2}.(1 + Cos2x)$$

$$Sin^3x = \frac{1}{4}.(3.Sinx - Sin3x) \qquad Cos^3x = \frac{1}{4}.(3.Cosx - Cos3x)$$

$$Sin(x \pm y) = Sinx.Cosy \pm Cosx.Siny \qquad Cos(x \pm y) = Cosx.Cosy \mp Sinx.Siny$$

$$Sinx.Siny = \frac{1}{2}[Cos(x-y) - Cos(x+y)] \qquad Cosx.Cosy = \frac{1}{2}[Cos(x-y) + Cos(x+y)]$$

$$Sinx.Cosy = \frac{1}{2}[Sin(x-y) + Sin(x+y)]$$

$$a.Cosx + bSiny = C.Cos(x+q) \qquad C = \sqrt{a^2 + b^2} \qquad q = \arctan(\frac{-b}{a})$$

A.3. Elementary Series:

$$Taylor\ Series: \qquad x(t) = x(0) + \frac{t}{1!}\dot{x}(0) + \frac{t^2}{2!}\ddot{x}(0) + \frac{t^3}{3!}\dddot{x}(0) + \cdots$$

$$Maclaurin\ Series: \quad x(t) = x(a) + \frac{t-a}{1!}\dot{x}(a) + \frac{(t-a)^2}{2!}\ddot{x}(a) + \frac{(t-a)^3}{3!}\dddot{x}(a) + \cdots$$

$Power\ Series:$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$Sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$Cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots$$

$$Tanx = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots \quad if\ x^2 < \frac{p^2}{4}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad if\ |x| < 1$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \binom{n}{k}x^k + \cdots + x^n$$
$$\approx 1 + nx \quad if\ |x| << 1$$

A.4. Sums and Geometric Series:

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2} \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \qquad \sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}$$

$$\sum_{k=0}^{N} a^k = \frac{a^{N+1}}{a-1} \quad if\ a \neq 1 \qquad \sum_{k=M}^{N} a^k = \frac{a^{N+1} - a^M}{a-1} \quad if\ a \neq 1$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{a-1} \quad if\ a \neq 1 \qquad \sum_{k=0}^{\infty} na^k = \frac{a}{(a-1)^2} \quad if\ a \neq 1$$

$$\sum_{k=m}^{\infty} a^k = \frac{a^m}{a-1} \quad if\ a \neq 1 \qquad \sum_{k=0}^{N} (\frac{a}{b})^k = \frac{a^{N+1} - b^{N+1}}{b^N(a-b)} \quad if\ a \neq b$$

A.5. Integrals:

$$\int_a^b x^n.dx = \frac{1}{n+1}x^{n+1}|_a^b \quad if\ n \neq -1$$

$$\int_a^b e^{cx}.dx = \frac{1}{c}e^{cx}|_a^b \qquad \int_a^b x.e^{cx}.dx = \frac{1}{c^2}e^{cx}(cx-1)|_a^b$$

$$\int_a^b Sin(cx).dx = -\frac{1}{c}Cos(cx)|_a^b \qquad \int_a^b Cos(cx).dx = \frac{1}{c}Sin(cx)|_a^b$$

$$\int_a^b x.Sin(cx).dx = \frac{1}{c^2}[Sin(cx) - cx.Cos(cx)]|_a^b$$

$$\int_a^b x.Cos(cx).dx = \frac{1}{c^2}[Cos(cx) + cx.Sin(cx)]|_a^b$$

$$\int_a^b e^{gx}Sin(cx).dx = \frac{e^{gx}}{g^2+c^2}[g.Sin(cx) - c.Cos(cx)]|_a^b$$

$$\int_a^b e^{gx}Cos(cx).dx = \frac{e^{gx}}{g^2+c^2}[g.Cos(cx) + c.Sin(cx)]|_a^b$$

$$\int_0^\infty x^n.e^{-ax}dx = \frac{n!}{a^{n+1}} \quad if\ n \geq 1, a > 0 \qquad \int_0^\infty e^{-a^2x^2}dx = \frac{1}{2a}\sqrt{p} \quad if\ a > 0$$

$$\int_0^\infty x^2.e^{-x^2}dx = \frac{1}{4}\sqrt{p} \qquad \int_0^\infty \frac{x^{m-1}}{1+x^n}dx = \frac{p/n}{Sin(mp/n)} \quad if\ n > m > 0$$

$$\int_0^\infty \frac{Sinx}{x}dx = \int_0^\infty \frac{Tanx}{x}dx = \frac{p}{2} \qquad \int_0^\infty Sinx^2dx = \int_0^\infty Cosx^2dx = \frac{1}{2}\sqrt{\frac{p}{2}}$$

$$\int_0^\infty \frac{Sin^2x}{x^2}dx = \frac{p}{2} \qquad \int_0^\infty \frac{Cosnx}{1+x^2}dx = \frac{p}{2}.e^{-|n|}$$

$$\int_0^\infty Sincx.dx = \int_0^\infty Sinc^2 x.dx = \frac{1}{2}$$

$$\int_0^\infty \frac{Sinx.Cosax}{x}.dx = \begin{cases} \frac{p}{2} & a^2 < 1 \\ \frac{p}{4} & a^2 = 1 \\ 0 & a^2 > 1 \end{cases}$$

$$\int_0^\infty e^{-ax} Sinx.dx = \frac{1}{1+a^2} \quad a > 0$$

$$\int_0^\infty e^{-ax} Cosx.dx = \frac{a}{1+a^2} \quad a > 0$$

Integration by Parts:

$$\int_a^b u(x)dv(x) = u(x) - v(x)\Big|_a^b - \int_a^b v(x).du(x)$$

Schwarz's Inequality:

$$|\int_a^b u(x).v(x)dx|^2 \leq \int_a^b | u(x)|^2 \, dx. \int_a^b | v(x)|^2 \, dx$$

A.6. Fourier Series and Fourier Transform:

Fourier Series of a well-behaving function as discussed in the body of the text is defined by:

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jnw_0 t} = a_0 + \sum_{n=1}^{\infty} a_n \cos(nw_0 t) + \sum_{n=1}^{\infty} b_n \sin(nw_0 t)$$
$$= c_0 + \sum_{n=1}^{\infty} c_n \cos(nw_0 t + \theta_n) \qquad \text{(A.1)}$$

and the coefficients can be evaluated from inverse relations:

D.C. Term = Time Average of $x(t)$:

$$a_0 \equiv c_0 = \frac{1}{T_0} \int_{t_0}^{t_0 + T_0} x(t)dt \qquad \text{(A.2)}$$

Cosine Terms with even symmetry:

$$a_n \equiv 2\Re e\{c_n\} = \frac{2}{T_0} \int_{t_0}^{t_0 + T_0} x(t) \cos(nw_0 t)\, dt \qquad \text{(A.3)}$$

Sine Terms with odd symmetry:

$$b_n \equiv -2 \operatorname{Im}\{c_n\} = \frac{2}{T_0} \int_{t_0}^{t_0 + T_0} x(t) \sin(nw_0 t)dt \qquad \text{(A.4)}$$

and the conversion formulas from one set to another:

$$a_0 = c_0 \qquad c_n = \sqrt{a_n^2 + b_n^2} \qquad \text{and} \qquad \theta_n = -\arctan(\frac{b_n}{a_n}) \qquad \text{(A.5)}$$

Fourier Transform: The Fourier transform of a well-behaving signal $x(t)$ is defined by:

$$X(w) = \int_{-\infty}^{\infty} x(t).e^{-jwt} dt \qquad \text{(A.6)}$$

and the Inverse Fourier Transform:

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(w).e^{jwt} dw \qquad \text{(A.7)}$$

Last two equations form the Fourier Transform Pair:

$x(t) \Leftrightarrow X(w)$ with

$X(w) = F\{x(t)\}$ and $x(t) = F^{-1}\{X(w)\}$

In general, $X(w)$ is a complex function of the real-valued independent variable w and it is written in terms of the magnitude and the phase:

$$X(w) = |X(w)|.e^{j\phi(w)} \quad \text{(A.8)}$$

Table A.1 Continuous Fourier Transform Properties

	Property	Time-Domain	Frequency-Domain
1	Superposition	$a.x_1(t) + b.x_2(t)$	$a.X_1(w) + b.X_2(w)$
2	Time Delay	$x(t - t_0)$	$X(w).e^{-jw\tau_0}$
3	Frequency-Translation	$x(t).e^{jw_0 t}$	$X(w - w_0)$
4	Scale Change	$x(kt)$	$\frac{1}{\|k\|}.X(\frac{w}{k})$
5	Time-Differentiation	$\frac{d^n}{dt^n}x(t)$	$(jw)^n.X(w)$
6	Time-Integration	$\int_{-\infty}^{t} x(\alpha).d\alpha$	$\frac{1}{jw}.X(w) + \pi.X(0).\delta(w)$
7	Duality	$X(t)$	$2\pi.x(-w)$
8	Parseval's Theorem	$\int_{-\infty}^{\infty} x(t).x^*(t)dt = \int_{-\infty}^{\infty} \|x(t)\|^2 dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(w).X^*(w)dw = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|X(w)\|^2 dw$
9	Time-Convolution	$x(t) * h(t) = h(t) * x(t)$	$X(w).H(w) = H(w).X(w)$
10	Time-Multiplication	$x(t).p(t)$	$\frac{1}{2\pi}X(w) * P(w)$
11	Analog Modulation	$x(t).Cos(W_c t)$	$\frac{1}{2}[X(w - W_0) + X(w + W_0)]$

Table A.2 Important Continuous Fourier Transform Pairs

	Description	Time-Domain	Frequency Domain	Constraint
1	Exponential	$e^{-at}.u(t)$	$1/(a+jw)$	$a>0$
2	Reflection	$e^{at}.u(-t)$	$1/(a-jw)$	$a>0$
3	Laplacian	$e^{-a\|t\|}$	$2a/(a^2+w^2)$	$a>0$
4	t^n * exponential	$t^n.e^{-at}.u(t)$	$n!/[(a+jw)^{n+1}]$	$a>0$
5	Delta Function	$\delta(t)$	1	
6	Generic Delta	$\delta(t-\tau_0)$	$e^{-jw.\tau_0}$	
7	D.C. Unity	1	$2\pi.\delta(w)$	
8	Single Harmonic	e^{jw_0t}	$2\pi.\delta(w-w_0)$	
9	Cosine	$Cos(w_0t)$	$p[d(w-w_0)+d(w+w_0)]$	
10	Sine	$Sin(w_0t)$	$jp[d(w+w_0)-d(w-w_0)]$	
11	Unit-Step	$u(t)$	$\pi.\delta(w)+1/jw$	
12	Signum Function	$Sgn(t)$	$2/jw$	
13	One-sided Cosine	$Cos(w_0t).u(t)$	$\frac{p}{2}[d(w-w_0)+d(w+w_0)]+\frac{jw}{w_0^2-w^2}$	
14	One-sided Sine	$Sin(w_0t).u(t)$	$\frac{p}{2j}[d(w-w_0)-d(w+w_0)]+\frac{w_0}{w_0^2-w^2}$	
15	Generic Oscillation	$e^{-at}.Cos(w_0t).u(t)$	$\frac{a+jw}{(a+jw)^2+w_0^2}$	$a>0$
16	Generic Oscillation	$e^{-at}.Sin(w_0t).u(t)$	$\frac{w_0}{(a+jw)^2+w_0^2}$	$a>0$
17	Rectangular Gate	$rect(t/t)=\Pi_t(t)$	$t.Sinc(wt/2p)=t.Sa(wt/2)$	
18	Sinc Function	$Sinc(w_0t/p)$	$(p/w_0).\Pi(w/2w_0)$	
19	Sinc Square	$Sinc^2(w_0t/p)$	$(p/w_0)\Delta(p/w_0)$	
20	Triangular	$\Delta(t/t_0)$	$Sinc^2(wt_0/2p)$	
21	Impulse Train	$\sum_{k=-\infty}^{\infty} d(t-kT_S)$	$T_S\sum_{k=-\infty}^{\infty} d(w-2pk/T_S)$	
22	Gaussian Signal	$e^{-t^2/2a^2}$	$s.\sqrt{2p}.e^{-s^2w^2/2}$	